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FREGE'S TWO NOTIONS OF "EXTENSION"[1]

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Abstract: The goal of this paper is to answer a question proposed by Richard Heck in the paper "Formal Arithmetic Before *Grundgesetze*". In that paper, Heck inquires as to the reasons why it took almost eight years for Frege to honor his promises of concluding his grandiose project of grounding mathematics on logic. Although Heck tried to answer his own question, we think that a more adequate philosophical discussion regarding Frege's delay can still be offered. This paper will try to fill in that gap by presenting what we understand was the central problem faced by Frege on *Die Grundlagen der Arithmetik*: the lack of a standard criterion to fix the meaning of identity propositions of mathematics. We believe that Frege's initial proposal of a dual character for "identity propositions" was the cause of all his problems in fixing his definition of numbers in *Die Grundlagen*. In the aforementioned eight-year period, Frege's challenge became that of finding a criterion capable of unifying his treatment of identities. In our account, the German philosopher finally decided to fill in this gap by providing a new construal of "extension", one which included some important refinements on his previous account of that notion. The new concept thus construed allowed Frege to unify his treatment of identity propositions by including in his system a universal and flexible criterion for deciding the truth of any identity proposition. Frege's new construal of "extension" was supported by his famous basic law V. So, our claim will be that Frege's resistance and doubts about the inclusion of axiom V as a logical law in his system were the primary cause of that delay.

Keywords: Frege, extension, identity.

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1. Introduction

In the foreword of *Grundgesetze der Arithmetik: Begriffsschriftlich Abgeleitet* (Gg, from now on) Frege tells us that it took longer than he expected to fulfill his promise of "carry[ing] out a project that [he] already had in mind at the time of [the] *Begriffsschrift* of the year of 1879". (2013, p. VIII) Frege's more definite effort to complete his foundational project went back to 1884 though, when he first announced it in *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung* (*Gl*, from now on). What is more surprising about Frege's huge effort is its time length. It took him almost eight years of intense work and, despite that, along those years he was forced to throw away an almost finished manuscript and start everything all over again. According to Richard Heck in the paper "Formal arithmetic before *Grundgesetze*", Frege himself never justified this attitude, and the only documentation we could have for understanding his predicament is lost.[4] (HECK, 2019, p. 3)

In our paper, we would like to step into the discussion with the following hypothesis. Frege's difficulties, which later proved insurmountable, came from a single source: the lack of a flexible and unified criterion for identity propositions. As we construed this great philosopher and mathematician's intellectual trajectory, in *Gl* Frege adopted only the ordinary meaning of the word "extension", borrowed from the one already available among mathematicians of his time. Furthermore, he thought at that time that "extensions" were just one

[4] As Heck tells us in his paper, there have been a few contributions suggesting what would be Frege's reasons for this extreme and puzzling attitude: one from himself, one from May (p. 10), from Sundholm (p.13), and from Boolos and Tappenden (p. 370). Cf. (HECK, 2019)

between other alternative criteria for deciding about the truth of mathematical identity propositions.

In *Begriffsschrift, eine der Arithmetischen nachgebildete Formelsprache des reinen Denkens* (*Bg*, from now on), Frege presented in §8 and 24 his first account of identity propositions. At that time identity propositions had for him a dual character, they were initially introduced as definitions, but, after this first logical moment, they were transformed into ordinary assertions of the system. As we are going to discuss in this paper, it was *Bg*'s construal of identities that were operational in *Gl*. But this old construal didn't work out quite as Frege had expected. In *Gl*, he realized that the dual character of identity propositions made their truth or falsehood dependent on their previous employment as definitions. As we are going to see in section III this dependency cuts off Frege's aim of being able to answer any question about identity between two randomly chosen objects from his universal domain. It consequently blocks his account of identity as a universal concept and his understanding that we have analytical identities with informational content.

One way out of Frege's dilemma was to adopt the ordinary concept of "extension", picked up from mathematician's practice, and to attribute to it the task of establishing true identities, especially those involving numbers within what he called "recognition-judgments" in a way that was independent of a preestablished and single definition. The ordinary notion of "extension" adopted by Frege in *Gl* though did not have independence from their intensional counterpart, the concepts, that he later discovered they should have, as we are going to discuss in section II.1. In the second period of his work which includes the three middle papers and Gg Frege solved this problem by presenting his own formal and rigorous definition of the notion of "extension". The definition was embedded in the cluster of stipulations that form Basic Law V and it was meant to provide a universal and flexible criterion for the truth of all identity propositions without any dependence on conceptual definitions.

The changes we claim Frege implemented in his earlier conception of "extension" are in perfect accord with his own view of the correct *desiderata* concerning any concept word included in his formal system. For him, any change in the characteristic notes of a concept already in use means a change in the very sense of the respective concept-word itself. As we are going to argue, the Gg concept of "extension" had acquired two new characteristic notes: 1) extensions had become logical objects *and were almost the unique inhabitants of* Frege's universal domain, besides the True and the False, of course; and 2) they became the representatives of all functions of any level at the zero-level, and as such, became capable of saturating any function in Frege's system.

In short, our goal will be to provide the following answer to Heck's question: Frege's had to introduce a fifth axiom in his system in order to solve his problems connected with the dual character of identity propositions. The solution consisted on the adoption of a new construal for the concept of "extension", endowed with two new characteristic notes. But as the law he had in mind could not be immediately taken to be a logical law without *unwelcome side effects*, during those eight years he must have tried other alternative solutions. Finally, he became convinced that basic law V was the only way out available for him to fill in that vacancy.

To provide our contribution to Heck's discussion, we will first deal with Frege's construal of "identity". We think that it was his original definition of identity and its dual character that led him to the need of transforming his system and providing a new construal for the ordinary concept of "extension". We also believe that, given Frege's philosophical assumptions, there was no other alternative at his disposal besides endowing extensions (or sets, or classes) with an ontological character. Our argumentative strategy will be to carry out this discussion in the context of some passages extracted from *Gl first*. Only after making Frege's problems in *Gl* clear will we move on to the riskier and more problematic scenario of *Gg*, just to introduce Frege's final solution.

I. Heck's proposal

According to Richard Heck in his paper "Formal Arithmetic Before *Grundgesetze*", the current exegetical debate on the periodization of Frege's work is actually very intense. He and those who worked with him on the project of furnishing us with a new translation of the *Grundgesetze* in English believe that a lot had changed in Frege's logical approach between *Gl* and *Gg*.[5] We quite agree with that. According to Heck, some explanation is missing though, concerning the reasons for Frege's desperate act.[6] We also agree with that

[5] The core team consists of Philip Ebert; Marcus Rossberg; Roy T. Cook (technical consultant); Crispin Wright (project coordinator). The consultants who provided invaluable feedback on our translation at our workshops: Michael Beaney; Gottfried Gabriel; Michael Hallett; Robert May; Eva Picardi; William Stirton; Christian Thiel; Kai Wehmeier, besides Richard Heck, of course.

[6] He writes: "Here's what I would like to understand: How, and in response to what pressures, Frege's logical doctrines evolve between *Begriffsschrift* and *Grundgesetze*. I propose to approach this

point. But Heck doesn't seem to believe that Frege's construal of "extension" had changed in any major aspect. In *Reading Frege's Grundgesetze* Heck claims that Frege's construal of "extension" was quite the same in *Gl* and *Gg*. His justification for this last belief can be found in two passages of his book: (2012, pp. 5, fn. 13) and (2012, p. 9). We disagree with him about this last belief.

Our claim will be then that Frege's new material, the one that took eight years to appear, mainly reflected a reformed position regarding the concept of "extensions". It was the new notion of "extension" that was operational in *Gg and Frege's three middle papers*. [7] So, the contribution of our paper to the discussion will be to argue that, contrary to Heck's claims (2012, p. 9), Frege's construal of "extension/value-ranges" presented in *Gg* engendered a new concept of "extension", which do involve value-ranges and which serve as grounds for a new definition of numbers also. The new concept of "extension" certainly could not be found in *Gl*. Due to its philosophical reformulation, as we have already pointed out in the introduction, Frege's new construal of "extension" had two additional characteristic notes that were not present in *Gl*. Let's discuss Heck's arguments to support his position, then.

In (2012, pp. 5, fn. 13), Heck offers us an interpretation of Frege's famous footnote 1 of §68 of *Gl* in support of

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question by exploring how the differences we have just reviewed might have affected Frege's attempts to derive the basic laws of arithmetic within logic. Why would those changes have forced him to abandon the work he had already 'nearly completed' in 1882 and to start afresh? If this was, indeed, why it took Frege so long to complete *Grundgesetze*, that cannot have been an easy decision for him but must have been one to which he felt forced." (HECK R. K., 2019, p. 3).

[7] (Function and concept, 1977), from now on: *FC, CO and SR* respectively.

his position. As he understood that footnote, Frege was not being rhetorical when he suggested that we could change the expression "extension of a concept" to the simpler expression "concept". In Heck's reconstruction, Frege had already been committed to that thesis. He suggests furthermore that the German philosopher had implicitly accepted the idea of interchanging the two expressions in the inverse direction also. Let us try to organize Heck's claims and evaluate them. Frege's original definition was: "The number which belongs to the concept *F* is the extension of the concept 'equal to the concept *F*'" (FREGE, The Foundations of Arithmetic, 1953, pp. 79-80) And Frege's suggestion in the footnote was literally the following: "I believe that for 'extension of the concept' we could write simply 'concept'." (p. 80). If we make the replacements, we will get: "*The number which belongs to the concept F is the ~~extension of the~~ concept 'equal to the concept F'*".

Heck's proposal, however, was that Frege was thinking also of allowing the reverse replacement as well. He thinks Frege would also agree to put back the phrase "extension of the concept" in place of the word "concept" getting: "the number of Fs [things which belong to the concept F] is the extension of the concept: extension of the concept *equinumerous with* F." (HECK, 2012, pp. 5, fn. 13) This time, the second instance of the word "concept" is also replaced by "the extension of...". Finally, Frege's formulation "*equal to*" is replaced by the expression "*equinumerous with*". According to Heck, this unconventional way to read Frege's footnote will offer us the same formulation presented in Gg. He writes, I quote: "This just is the *Grundgesetze* definition, modulo the switch from extensions to value-ranges." (2012, pp. 5, fn. 13)

On page 9, Heck resumes his position concerning Frege's construal of "extensions" and gives the following explanation:

> Since extensions are just a special sort of value-range, and application is a generalization of membership, it is therefore easy to restate the definition of numbers given in *Die Grundlagen* in the new framework of *Grundgesetze*: Numbers are still defined as extensions in *Grundgesetze*, indeed, *as the very same extensions as in Die Grundlagen*; [...] (HECK R. G., Reading Frege's Grundgesetze, 2012, p. 9) [my emphasis]

Although we disagree with Heck for philosophical reasons, in the literature there are also those who also disagree with him, but for different and more technical ones. In an otherwise highly enthusiastic review of Heck's book, Marcus Rossberg, one of the editors and translators of the *Grundgesetze* to English, discords with Heck.[8] In his opinion, although Heck's claim "is inconsequential for the remainder of his arguments", it is nevertheless, "not quite right and, moreover, potentially problematic". In his review Rossberg offers two reasons for his disagreement. First, he thinks one cannot simply reduce value-ranges to extensions. The universal domain of extensions in Gg must also include value-ranges, those that are the result of functions which are not concepts.[9]

It is out of the scope of this paper to deal in more detail about this debate. We believe that Schirn, Cook, and Rossberg are right in disagreeing with Heck about this point,

[8] (ROSSBERG, 2012 (2014)).

[9] Rossberg bases his disagreement on some arguments given by: Roy Cook (2014) in his review of Patricia's *Frege's conception of Logic* book; and in Matthias Schirn's paper (SCHRIN, 1990).

but in our opinion, they argued exclusively from a technical point of view. None of their arguments addresses the true source of Frege's problems and, so, they could be overcome by some other recourse on technicalities. In our opinion, Frege was primarily a great philosopher and we do believe that one can only comprehend his position and make amends to his system if all his philosophical assumptions were taken into consideration.

As we are going to show at the last section of this paper, even Russell's paradox could be overcome by means of another technicality, a piecemeal interpretation of "identity", as Russell himself had proposed to Frege. Nevertheless, the elder's answer to that was also philosophical and he refused Russell's proposal wholeheartedly, as we will discuss in that section. We believe likewise that Frege's philosophical worries, those which were expressed in footnote 1, reflected his desire for defining identity universally for the entire domain. We believe that Frege was primarily preoccupied with choosing the right criteria to perform this task, as we are going to argue next.[10]

II. Frege's initial ideas about definitions and identity

II.1 Contextual definition of numbers or the fruitfulness of analysis in *Bg* and *Gl.*

As we know, in 1879 Frege wrote the book *Begriffsschrift: eine der aritmetischen nachgebildete Formelsprache des reinen Denkens (Bg, from now on).* It was his first attempt to create a

[10] This problem is frequently referred to as the Cesar problem. We decided not to use this nomenclature, for we are not so sure this is the same problem we are trying to encapsulate here.

universal language. He called this language a "concept-script", a script of concepts. In this first effort to regiment ordinary language and build his concept-script, Frege used just one single construal of "sense" for the entire language, what he called "conceptual content". Every part of any proposition would have a conceptual content of its own, but only as a part of the conceptual content of a whole proposition. This idea was enunciated by him in the last paragraphs of *Gl's* introduction as one of the three fundamental principles to be followed throughout the book: "*never to ask for the meaning of a word in isolation, but only in the context of a proposition*". (1953, p. xxii)

Frege's conceptual-script included also rules of grammar, of course. This means that he offered a detailed project on how to understand the internal structure of any sentence. Frege's semantical/grammatical approach did not follow either natural languages or formalized languages as we currently understand them. Actually, we could say that he had just one single division: logical subjects and logical predicates. But, as we are going to see, this division could also be reduced to another more economical one: conceptual contents, on one side, and the universal predicate "is a fact", on the other.

Another important distinction between the Fregian approach and the other logical approaches known so far is that in Frege's early account, the conceptual content could be construed as performing one between two distinct roles, depending on how one analyzes the proposition which contains it. The first analytical stage involves choosing one or more parts of the sentence to be considered as the subject matter[11] and the second stage is to let the rest be the functional part. Therefore, the attribution of a semantical role to any fragment of a sentence would depend on how the analytical process was implemented, i.e., on how one chose to answer the following two questions: (1) "What are you talking about?" (2) "What is being said about it?".

As a result of Frege's early liberal approach towards the semantical role of any expression, the determination of its sense would be achieved only as a result of what he called his "fertile method of analysis". That is to say, it would depend on the result of pulling out the specific expression from its place and entrusting it to one of the two roles: either it would be taken as an argument, the subject matter of that sentence (about which something is asserted), or it would be taken as the functional part (what is affirmed about this same subject matter).[12] But the most important aspect of the strategy adopted by Frege is that undetached, i.e., taken out of the context of a whole sentence, neither of those sentential parts could be said to perform any pre-established semantical role.

We could concede that this freedom of choice as to how to implement a propositional analysis presented in the last paragraph was an idiosyncrasy of Frege's initial ideas, at least when evaluated from our contemporary point of view. We could even say that his position in this *period before* Gg's involved a sort of top-down semantics. And we could also add

[11] In case of relations, one could choose a pair or even a bigger sequence of names.

[12] Following Chateaubriand, we will call the part of the proposition that answers the question "What is being talked about?" the "logical subject" and the rest of the proposition, the part that answers the question "what is affirmed of this?" the "logical predicate". As he himself points out in his book, this distinction only became clearer after Frege's article "On Sense and Reference". (CHATEAUBRIAND, 2001, p. 240-244, chap. 2).

that this approach was a direct consequence of applying the context principle. On this specific point, we do agree with Heck: the same liberty involved in considering any part of the sentence as the argument or as the functional part did not remain operative either in *Gg* or in his "three middle papers". In those latter works, Frege adopted a rigid way to assign semantical roles to parts of sentences as we are going to discuss further on. (HECK & MAY, 2013, p. 5)

All those previous commentaries seem to point to a complete relativization of the domain, an almost absent notion of objects as primary things. We think that all those ideas of how to see Frege's initial work do not change a central point, though. He still thinks that true thoughts have an objective character and that in mathematics they should speak about numbers taken as objects.

The same process of extracting the conceptual content of an expression from the whole conceptual content of the sentence could be applied, of course, to numerical propositions. In *Gl*, therefore, the sense of any numerical expression would have, as it were, to be "carved up" from the mathematical identities to which it originally belonged: "We carve up the content in a way different from the original way, and this yields us a new concept." (FREGE, The Foundations of Arithmetic, 1953, p. 75. §64)

Frege also applied this same methodological approach to the derivations of all new theorems. We will refer to Frege's method of analysis as the "fertile or fruitful analysis". In paragraph 88, Frege mentions the adjective "fruitful" in connection with mathematical definitions.

> [Kant] seems to think of concepts as defined by giving a simple list of characteristics in no special order; but of all ways of forming concepts, that is one of the least fruitful. If we look through the definitions given in the course of this book, we shall scarcely find one that is of this description. The same is true of the really *fruitful* definitions in mathematics, such as that of the continuity of a function. What we find in these is not a simple list of characteristics; every element in the definition is intimately, I might almost say organically, connected with the others. [my emphasis] (FREGE, The Foundations of Arithmetic, 1953, p. §88)

In this passage, Frege was criticizing Kant for not arranging the characteristic notes of a concept in any determined and structured way. In the sequence, Frege suggests a metaphor:

> A geometrical illustration will make the distinction clear to intuition. If we represent the concepts (or their extensions) by figures or areas in a plane, then the concept defined by a simple list of characteristics corresponds to the area common to all the areas representing the defining characteristics; it is enclosed by segments of their boundary lines. With a definition like this, therefore, what we do in terms of our illustration is to use the lines already given in a new way for the purpose of demarcating an area. Nothing essentially new, however, emerges in the process. But the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. (FREGE, The Foundations of Arithmetic, 1953, p. §89)

The image of a map helps us to understand how new concepts should be formed, according to Frege. They would be the result of rearranging the older ones' frontiers. According to Frege, this implies reorganizing the characteristic notes of the original concepts, taken from more than one premise, in a new inferential order of logical priority. The new concept's definition would then be the list of all those older characteristic notes, now rearranged and logically connected within a new structure. Finally, according to Frege, the result would constitute a proof of "some propositions, which were not contained in any one of them alone". (FREGE, 1953, p. §89)

The idea of "fruitfulness" and the suggestion of a method of proof embedded in Frege's analytical method were both based on his belief that for any sentence to be true it ought to express an objective thought concerning reality and, therefore, we can never create anything "completely new". Our job as philosophers of mathematics should only be to choose items from the list of characteristic notes already contained in the concepts under investigation and rearrange them in a novel way. Frege explains his idea of "fruitfulness" with a second metaphor: "[The conclusions we draw from our rearrangement of frontiers] are contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house." (FREGE, 1953, pp. 101, §88) Once again, Frege reinforced in this passage the idea that one cannot introduce new concepts and new objects arbitrarily, "out of nothing", as it were. What one could do is use a new concept expression and just attach to it those older characteristic notes, now rearranged within the new logical structure. So, according to Frege, the "new concepts" were already potentially contained in our old concepts and were carved up from them.

The generality achieved by Frege's method was indeed a liberation from the traditional division between subject-concept and predicate-concept, perpetuated for many centuries by the logical-philosophical tradition since Aristotle.

(DUMMETT, 1973, p. 8) In that traditional logic, one could have only monadic predicates. With Frege's method, though, one could form concepts and relations of first, second, or even higher logical levels which had never been formed before.

II. 2 A geometrical example

In §§ 63-68, Frege uses his fruitful method of analysis to achieve his primary goal: to define numbers as objective referents of mathematical propositions. He began tough with a geometrical example: "line *a* is parallel to line *b*". In this example, a relationship of "parallelism" holds between line *a* and line *b*. After proposing the example, Frege asked himself if the concept of parallelism could ever be converted into an identity between two objects. His answer was to use the idea of fertility presented in the last section:

> ... through removing what is specific in the content of the former [...is parallel to...] and dividing it between *a* and *b*. *We carve up the content in a way different from the original way*, and this yields us a new concept. [my emphasis] (1953, p. 75)

In this passage, the conceptual content of the predicate "... is parallel to ..." was split between the two *relata*. The result was that one "*carve up the content in a way different from the original way*" and transform the "parallel relation" into the new concept of "direction" plus the identity relation. We could then say that *a* and *b* are identical with respect to "direction". In the following passage, Frege provided more details about how this transformation should take place:

> The definite article indicates the following. For me, a concept is a possible predicate of a singular assertible content [using the judgment bar], an object, a possible subject of such an assertion. If we see in the sentence, "The direction of the telescope axis is the same as the direction of the earth axis", *the direction of the telescope axis* as a subject, then the predicate would be "equal to the direction of the earth axis". This last would be a concept. But "the direction of the axis of earth" is a part of the predicate; [so it could be taken as] an object because it can also be made a subject. (1953, pp. 77, footnote 2)

In this footnote, Frege mentions two important instruments to accomplish his goal of transforming a particular relationship into a property applied to each *relatum* plus the identity relation. They are: (1) the nominalization of any conceptual content; and (2) the usage of the "definite article". (1879, pp. 12, §3) In *Bg*, Frege had already adopted the resource of nominalizing the conceptual content of a sentence adding to all those "names of contents" the universal predicate "is a fact". In that work, he has also adopted the usage of the "definite article" as a theoretical expedient to form those "names". In fact, although Frege had not yet given to it a formalized version, presented just in Gg, the definite article "operator" had already an informal content that was used in practice within his formal concept script: "there exists one and only one thing which has the mentioned singular assertible conceptual content". (1953, pp. 87, ft.1) With those two instruments in hands, Frege could always turn some conceptual contents, those that could be rephrased as "the circumstance that ...", into the subjects of the universal predicate "is a fact". The result is the transformation of all the conceptual contents (including those that are not part of the subject) into names.

Returning now to the example concerning parallelism, let's revisit it with those two new ingredients in mind – the definite article and the nominalization of conceptual content – and see how they help Frege in the achievement of a final definition of "direction". In that example, the original conceptual content "The direction of the telescope axis is the same as the direction of the earth axis" was analyzed first as: (1) a nominal part "The direction of the telescope axis"; and (2) a functional part "... is the same as the direction of the earth axis". Then, further deepening the analytical process, we could apply the definite article to part of the content of the functional part – "...is the same as *the direction of the earth axis*" – and obtained the name "*the direction of the earth axis*". The residue of this second analytic deepening was the assertion "*...is the same as...*". This residue is the identity relation[13] and its components content analyzed away, the conceptual content of a new name. The new name was the name of the new upper-level function/concept "being a direction". This second order concept can now unify lines into sets of "lines that have the same direction".

When reviewing Frege's examples of parallelism, it is also worth remembering that he had begun his entire project trying to consolidate the following idea: "to construct the content of a judgment which can be taken as an identity such that each side of it is a number." (1953, pp. 74, §63) In the parallelism example, we achieved names for directions on both sides of the identity sign. In the case of numbers, his

[13] According to Frege's definition of identity present in the next section.

final goal, Frege intended to construe a concept that could be used to name the same number at each side of the identity sign too. Let's not forget also that this assertion must be in many cases informative. We can thus conclude that, at the time of *Gl*, the fertile method of analysis, as well as the nominalization and the help of the definite article were all the tools Frege believed were needed to begin his search for the adequate definition of "number" and to prove that arithmetic is composed of analytical truths, which nevertheless expands our knowledge. However, this belief proved to be false later on, as will see in section IV.

III. Frege's definition of Identity

III.1 Identity in *Bg*

So far, we have discussed how Frege intended to use his analytic method and the definite article in his search for an adequate conceptual content for identity propositions involving numbers. Let us now deal with the concept of "identity" itself.

Identity propositions have always presented a challenge to Frege. In §62 of *Gl* he was already looking for a general criterion of "numerical identity" and in §104 he announced that he needed to "fix the sense of a recognition-judgment for the case of the new numbers". However, if we go further back in his work, we find him dealing with his first effort to regulate the enigmatic and dual character of identity propositions in §8 of *Bg*.

In this first attempt to create a formula language, Frege admitted that he was *forced* to prescribe a "dual character" to all formulas expressing identities. (1879, pp. 55, §24) He explained what he meant by "dual character" right at the beginning of *Bg*, in (1879, pp. 20, § 8) as we've said. He began there with the first character of identities, that of "definitions". This aspect concerns the introduction of new signs for new conceptual contents in language. In these logically primary cases, identities should be taken as dealing with the signs taken by themselves. In Frege's own words: "they suddenly display their own selves". (1879, pp. 20, § 8) When used according to this first approach, identities would not be predications, nor assertions, but definitions or rules. This first aspect is indicated by the presence of two vertical bars before the horizontal line of content, instead of just one. The notational convention specifies that what follows the horizontal line should be taken as a kind of rule, for they would have a normative and not an assertive character.

Two comments are relevant concerning this first character of identities prescribed by Frege. The first is that at this time for him every linguistic element had just a conceptual content, be it a name or a whole sentence, for he had not divided it into sense and reference yet. So, if one considers this undivided content, the advice is to be careful as to what was been presented by a linguistic expression, for it could be, either a sense, a single object or even a whole fact.

The second comment is about the interpretation of §8 of *Bg*. We think that when exploring this first aspect of identities Frege was not suggesting that we just stipulate an arbitrary connection between sing and conceptual content. That is to say, for him, in this first usage, although identities would not be assertions, they would not be like arbitrary rules for the correct use of signs either. If they were, then they would have to be made only once, for each word, and should be taken as the single way any content, be it a concept or an object, could be introduced. That was not what Frege wanted.[14] Quite on the contrary, in *Bg* Frege was trying to circumscribe the fact that those new contents, the results of an analytical procedure, or else of an alternative description of the same thing, could be the content of a new name of that same entity, a new way to present the same entity. In the sequence, Frege gives his geometrical example of a fixed point in a circle:

> To each of these ways of determining the point there corresponds a particular name. Hence the need for a sign for identity of content rests upon the following consideration: the same content can be completely determined in different ways; but that in a particular case two ways of determining it really yield the same result is the content of a judgment. (1879, p. 21)

It was crucial for Frege that this new content consisted of a new name and represents a new way of presenting the same "conceptual content". The ambiguous mention of "conceptual content" in this passage is confusing though, as we've already anticipated. The new "conceptual content" could be either a reference or two different presentations of the same referent. We will talk more about this ambiguity soon. Another critical point is that Frege didn't see any alternative for the introduction of this new content at that time then to ascribe it to a new expression via a new definition:

[14] Frege called the first use of identity with two vertical bars definitions. But as we are going to explain later, they were not exactly what is traditionally called a definition, because one can make several of them about the same object. This is essential if one wishes to identify the same object by different modes of presentation, as we will see later.

"Before this judgment can be made, two distinct names, corresponding to the two ways of determining the content, must be assigned to what these ways determine." (1879, p. 21)

After the introduction of new signs and contents by identities taken as "definitions" or "rules", a strange twist occurred in their nature. Once established that those two different "names" should be used to present the same object, one of the two vertical bars could be dropped, and those *definitions* were thus "turned into judgments". (1879, p. 21) From that moment on they assumed the role of actually referring to the same conceptual content, or "to yield the same result". (1879, p. 21) So, those sentences that were just definitions/rules suddenly become judgments with content. Their content would be that one has actually more than one *conceptual* way to present the same *conceptual content*.

In § 24, Frege returned to the subject and says about proposition 69:

> This proposition differs from the judgments considered up to now in that it contains signs that have not been defined before; [themselves] give the definition. It does not say: 'the right side of the equation has the same content as the left', but 'it has to have the same content'." (1879, pp. 55, §24)

In this passage, Frege gave the example of proposition (69) that introduces a new sign for the higher order property which asserts that property *F* is hereditary in the *f*-sequence. He emphasized also the normative character of this first definitional proposition. According to him, it determines a new content, and therefore, although proposition 69 should not be taken as a judgment, it...

> … could be immediately transformed into one, for, once the meaning of the new signs is specified, it must remain fixed, and therefore formula (69) also holds as a judgment, but as an analytic one, since it only makes apparent again what was put into the new signs. (1879, pp. 55, §24)

The transformation should be made by dropping one of the vertical bars, as we have said. From that point onwards anyone could use this identity to assert that both sides present the same *content*. As we quote before, in (1879, pp. 21, §8) Frege had said that "two ways of determining it really yield the same result is the content of a judgment". In this passage, we can notice the ambiguity into which Frege was driven by this new element, the idea of "*two ways of determining … the same result*". It seems strange that in the first construal of identities, we had two expressions naming the same content, and in the second one those two expressions stand for two different ways (two different conceptual contents) that should determine the *same … "content"*! Besides, this "something" that should be fixed was called by Frege, either an object or a content taken as the object, depending on the context.

It is now easy to see how problematic was Frege's first attempt to explain identities. In his own words, what he called "definitions" should deal just with "names and not with contents". But, also according to him, they should establish the very fact that those two names, which were the result of the nominalization of two different conceptual contents, should be employed henceforth to present the same "content".

III.2 Identity in *Gl*

III.2.1 The request of generality and Leibniz's definition

An essential aspect of identity propositions for Frege was their universality. For the philosopher, there cannot be a special kind of identity created exclusively for numbers. As he formulated the issue in the following passage from *Gl*, a more general criterion for when two presentations were indeed two different ways of introducing the same object was an inevitable theoretical requirement. The generality of this requirement was not to be restricted to the type of object being tested. These could be numbers or any other type of object whatsoever.

> It is not only among numbers that the relationship of identity is found. From which it seems to follow that we ought not to define it specially for the case of numbers. We should expect the concept of identity to have been fixed first, and that then, from it together with the concept of Number, it must be possible to deduce when Numbers are identical with one another, without there being need for this purpose of a special definition of numerical identity as well. (FREGE, The Foundations of Arithmetic, 1953, pp. 74, §63)

In this passage, Frege confirmed what he thought should be the correct order in the way to define numbers: first, we need to say what it means for any two objects to be identical, and just after we've done that, we then could look for the specific explanation of numerical identities.

Frege's form of argument is frequently encountered in philosophy. It is called *the petitio principii* fallacy or *the fallacy of the circular argument*. It says that definitions cannot include in the *definiens* the term to be defined in the *definiendum*, otherwise, we would be presupposing the very concept that we aimed to define in the first place, in this case, the concept of "number". A claim of circularity surely makes any definition useless and unreliable.

> Our aim is to construct the content of a judgment which can be taken as an identity such that each side of it is a number. We are therefore proposing not to define identity specially for this case, but to use the concept of identity, taken as already known, as a means for arriving at that which is to be regarded as being identical. (FREGE, 1953, pp. 74,§63)

That is to say since numbers should be objects akin to any other object of the domain, Frege must look for a general criterion that could decide over the identity between any two possible candidates picked from the entire universal domain. Furthermore, a general account of identity for any object was still required before one can turn to the specific explanation of numerical identities, for we cannot use the concept of "number" in our definition of identity between numbers at risk of incurring in circularity: "We should expect the concept of identity to have been fixed first, and that then, from it together with the concept of Number, it must be possible to deduce when Numbers are identical." (1953, p. 74)

Frege had already given an explanation of "identity propositions" in *Bg* §8, and he was not proposing to change it in *Gl.* What he needed then was not a new explanation of the roles of identities, but an external way to decide when to things are the same. His proposal in *Gl* §65 was to take Leibniz's definition of identity as a *desideratum* to guide him in his search.

Now Leibniz's definition is as follows: "Things are the same as each other, of which one can be substituted for the other without loss of truth". This I propose to adopt as my own definition of identity. (1953, p. 76) In Leibniz's definition the expression "the same" was the focus of Frege's commentaries. He stipulated then that:

> Whether we use "the same", as LEIBNIZ does, or "identical'', is not of any importance. "The same" may indeed be thought to refer to complete agreement in all respects, "identical" only to agreement in this respect or that; but we can adopt a form of expression such that this distinction vanishes. For example, instead of "the segments are identical in length", we can say "the length of the segments is identical" or "the same". (1953, p. 76)

So, according to Frege, "*the same*" would mean having all aspects, or properties, in common, and "*being identical in …*" would mean that two singulars possess "*the same*" property.

The obtainment of a unique criterion for identity propositions was not just a philosophically unimportant matter for Frege, but something of a crucial nature regarding the foundational character of his philosophical project. Frege made strange claims in *Gl* about the need of applying identity universally:

> [...] our definition affords us a means of recognizing this object as the same again, in case it should happen to crop up in some other guise, say as the direction of b. But this means does not provide for all cases. It will not, for instance, decide for us whether England is the same as the direction of the Earth's axis – if I may be forgiven an example which looks nonsensical. (FREGE, The Foundations of Arithmetic, 1953, p. §66)

What seems nonsensical in Frege's assertion is that one is invited to compare completely different things like a country and the Earth's axis. Of course, nobody would have difficulty distinguishing between those two "empirical" objects, as Frege himself recognized. But he had a good reason for giving such a bizarre example. His concern was to make clear to what extent the universal character of identity should be maintained.

III.2.2 The problematic dual aspect of identities persists in *Gl.*

Let's now return to the parallelism example. In section II.2, we've followed step-by-step Frege's usage of his method of analysis in the geometrical example of two parallel lines. We explained how he extracted the conceptual concept from each side of the original assertion using the nominalization method and the application of definite article. Next, we present how Frege carved out a new concept and obtained the statement that the directions of the two lines were identical. Now we can see the same example in the new light of *Gl*'s improvement: Frege's general and *a priori* definition of identity borrowed from Leibniz. After settling his universal definition of identity Frege wanted to prove that it could be applied to any identity context, i.e., to recognition-judgments asserted about any two names whatsoever.

Resuming our discussion of his example, after denning that England could be the same as the direction of the Earth's axis for obvious and empirical reasons, Frege goes on and says that he has to decide, for any new and unknown thing *q*, if "the direction of *a* is identical with *q*" or if "*q* is identical with the direction of *b*". Confronted with this new challenge, Frege concluded that what he was missing was the concept of "direction" in order to stablish if *q* is a "direction".

In §67 of *Gl*, Frege revisited his old definition from §8 of *Bg* with the parallelism example in mind. He was trying to test it against this new question: Is *q* a direction? One of the options Frege considered was "*q* is a direction, if there is a line *b* whose direction is *q*." (1953, pp. 78, §66) However, he goes on and say that "then we have obviously come round in a circle.", i.e., that this attempt of defining the concept of "direction" involved some circularity. What he really needed was to decide about the truth or falsehood of any identity containing *q* without resorting to the explanation that it was a direction, because it was identical with another direction. According to him, "we should have to know already in every case whether the proposition "*q* is identical with the direction of *b*" was to be affirmed or denied. (1953, pp. 78, §66)

Or else we should have to know already what a direction is, without resorting to its identity with another thing that is already known to be a direction. In the beginning of §67, Frege raised the following alternative possible reading of the first character of identities, their "definitional role":

> If we were to try saying: *q* is a direction if *it is introduced by means of the definition set out above*, then one *would* be treating the way in which the object *q* is introduced as a property of *q*, which it is not. The definition of an object does not, as such, really assert anything about the object, but only lays down the meaning of a symbol. After this has been done, the definition transforms itself into a judgment, which does assert about the object; but now it no longer introduces the object, it is exactly on a level with other assertions made about it. (1953, pp. 78, §67) [my translation]

In this passage, we can notice that it was just in *Gl* that Frege began to realize how problematic the distinction between those two roles of identity could end up being. When assuming a "definitional role", identities could be interpreted as a unique act of definition, the sole way of introducing an object. But if this path was chosen, then, according to our interpretation, Frege was saying that the definition could be confused with a property of that object. Frege is categorical: it is not. He then realized that a simple negation would not be sufficient. Anyone can make that confusion or simply assume that the definition is a property of the object, the property of "being introduced by definition *D*", for example, then one would still be treating the fact that the object *q* was introduced by definition *D* as a property of *q*. Concerning this alternative, Frege complains:

> If, moreover, we were to adopt this way out, *we should have to be presupposing that an object can only be given in one single way*; for otherwise it would not follow, from the fact that *q was* not introduced by means of our definition, that it *could* not have been introduced by means of it. (1953, pp. 78, §67) [my emphasis]

The usage of double negation makes this passage especially difficult to be grasped. Let's try to understand what Frege was then saying. To begin with, the best scenario for him would be the one where the dual character of identities should directly imply that the same object could be presented *in more than one way*. What seems to be worrying Frege in this passage is that, if someone follows his own prescription of a first "definitional role" of identities but treats the very way in which the object *q* is introduced as a property of *q*, thus applying to *q* the property of "being introduced by definition *D*", then this person would be treating the last property as the sole criterion for something to be a "direction". According to Frege's reasoning, this conclusion would be unavoidable to that person because otherwise, if *q* were not introduced by *D* and it could not be introduced by D, then it would have to be introduced by another definition *D'*. In this hypothetical case one would have to evoke another criterion for stablishing when those two objects, being introduced by two different definitions, *D* and *D*', were both directions. This way out obviously would lead that person to an infinity regress. But someone who adopts it would, of course, to avoid this vicious regress. So, that person would have to accept the conclusion that: *q* was a direction if, *and only if*, it was introduced exclusively by means of the first definition *D*.

In the somewhat obscure passage quoted above, Frege realized that in the scenario where this wrong interpretation was not ruled out his whole logicist project would doomed to failure.

> All identities would then amount simply to this, that whatever is given to us in the same way is to be reckoned as the same. This, however, is a principle so obvious and so sterile as not to be worth stating. [...] Why is it, after all, that we are able to make use of identities with such significant results in such diverse fields? Surely it is rather because we are able to recognize something as the same again even although it is given in a different way. (1953, p. 78)

In this subsequent passage Frege complained that a wrong interpretation of the "definitional usage" of identities would turn all identities into platitudes and therefore would completely miss the goal of providing for mathematical propositions to be analytic but also informative. The idea of having only one way to introduce a new object into the domain would put Frege in the same position as Kant: all analytical judgments would be sterile. This conclusion, of course, would completely defeat Frege's logicist project.

Now, let's discuss the other horn of Frege's dilemma. We think that, as much as Frege was willing to aloud as many ways was to introduce an object via a definitional identity as one needed, without any single one of them to be taken as a property of the object and as the unique way of identifying that object, he would still have to face just two options. Either he would say that *q* is the object introduced just by the definition of "direction *D*", or that *q* would also be a direction, but it would have been introduced by other properties. But, if the first case was tautological and was thus rejected by him, the second didn't seems to provide for the cases where *q* was not a direction, for if *q* was not a direction, how could one establish if *q* was identical with the direction of *b* or not? Frege concluded then that he must find a new criterion that works for all cases.

As much as we understand that Frege's first *desideratum* was to allow for multiple ways to recognize again a thing as being the same, a lot of questions still seemed to lack appropriate answers. For example, how could one compare numbers with objects that happen not to be numbers, say, physical objects, for instance? Or, even if we stay within the context of numbers and the equinumerosity property, what about numbers that could not be segregated by the equinumerosity property in the first place, like irrational sequences or even "infinitesimals", for example?[15]

IV. Frege's alternative method involved extensions

In the paragraphs succeeding §67, Frege decided to suggest a new criterion that could avoid all those questions. Frege's persistent attitude reflects his belief that it must be possible to find a non-circular definition of "direction" or of "numbers", one that would avoid both circularity and also the infinite regress resulting from multiple ways of defining the same object. This alternative method identifies the new second order concept "the direction of line *a*" with its source, the extension of the concept "...parallel to line *a*". As was already settled by Frege's fertile method of analysis, every line that is parallel to line *a* would have the same direction as it. We then get the following criteria:

> the direction of line *a* is the extension of the concept "parallel to line *a*";
> the shape of triangle f is the extension of the concept "similar to triangle *t*".

Next, Frege tried to extend this same idea to his definition of "numbers".

> To apply this to our own case of Number, we must substitute for lines or triangles concepts, and for parallelism or similarity the possibility of correlating one to one the objects which fall

[15] In " (Infinitesimals, Magnitudes, and Definition in Frege, 2019, p. 253), Tappenden gives an example of one of the profound insights depended on identifying objects presented in drastically different ways, even crossing disciplinary boundaries.

> under the one concept with those which fall under the other. For brevity, I shall, when this condition is satisfied, speak of the concept *F* being *equal* to the concept G; [...] (1953, p. 79)

The new extensional method proposed by Frege had one advantage over the definitional one for it excluded de rigidity of definitions, which would require but a single way of introducing an object.

As we saw, the new method suggested by Frege, when applied to lines or triangles, uses the extensions of the concepts of "parallelism" or "similarity", each of them corresponding to its own extensions. In the case of numbers, Frege's new method uses Hume's principle instead, i.e., the concept of "equinumerousity". Hume's principle is just the logical possibility of achieving a correlation one-to-one between the elements of any two concepts' extensions. Frege concluded that: "when the objects falling under the concepts F and G are correlated with each other by a correlation [... that] has to be one-one." (FREGE, The Foundations of Arithmetic, 1953) We could say that "the concept F is equal to the concept G." (FREGE, The Foundations of Arithmetic, 1953) and that "the Number which belongs to the concept F is the extension of the concept 'equal to the concept F'" . (1953, pp. 79-80)

In this passage, instead of "the extension of the concept 'parallel to line a'" we have this new extensional criterion, "the extension of the concept 'equal to the concept F'" that works for numbers. If a number belongs to some concept F, then it also belongs to any other concept whose extension is equal to it.

This other method was an improvement over definitions, but it also brought in new challenges, though. The first one is that the new way of using identities as assertions about concept extensions would be somewhat strange, for identity is a relation between objects and not between concepts. Frege suggested the word "equal" instead of "identical" to emphasize the idea that the concepts have the equal number of elements, but do not necessarily have to express the same content.[16] In footnote 1 of page 80, already mentioned in section I, Frege asked himself if those two statements "the extension of the concept equal to the concept F" and "the concept equal to the concept F" would be equivalent. This is a difficulty that Frege only solved in Gg, much later on.[17]

The second difficulty brought in by Frege's new method is that, for each new second-order concept, we will have to stipulate de correspondent extension, one which could be used as a criterion. But again, for Frege what one needed was a universal criterion but what we end up having is one criterion for numbers and another for parallel lines, as we are going to see in next section.

Summing up our conclusions up until now, Frege's definition of "identity of contents" presented in *Bg* brought undesirable consequences to his construal of "number identities" as "analytical but informative" in G*l*. On the one hand,

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[16] According to a footnote written by J. L. Austin the translator of *The foundations of Arithmetic* about the use of word "equal" between the two concepts: *Gleichzablig* is an invented word, literally "identinumerate" or "tautarithmic"; but these are too clumsy for constant use. Other translators have used "equinumerous"; "equinumerate" would be better. He also adds that later writers have used "similar" in this connection (but as a predicate of "class" not of "concept"). (1953, pp. 79, footnote)

[17] We will devote an entire to deal with this single footnote later (section IV.2).

if a number was introduced by a definition, we could only recognize any other object as the same if it were introduced by that same definition. On the other hand, if we could introduce the same number by a new property, we would be unable to decide if they were really the same, as we've argued before. In this section we have also shown how Frege's suggestion of moving from conceptual definitions to the extensions of the concepts involved a new to solve these problems. In the next section we will explore those two difficulties indicated in the last paragraph.

V. Frege's problems in *Gl*

V.1 Frege's concept of "extensions" in *Gl*

In the last section, we have shown that the dual character of identities introduced in *Bg* brought problems to Frege's account of identity in the case of parallel lines and in the case of numbers. Finally, we began to explore Frege's new method, which uses extensions instead of definitions and, in the case of numbers, uses the one-to-one relational property (Hume's principle) to compare concepts' extensions in general and thus identify when two numbers are the same. As we saw in last section, the new method of moving from the second order concept of "direction" or of "equinumerousy" to the extension of *those* concepts was contemplated by Frege as a way to decide between something unknown and a direction *a*, or between something unknown and a number. We want to deal now with Frege's doubts regarding that method as the more adequate for deciding over recognition-judgments of identity applied universally.

Frege's new method involved directly the extensions of each concept of his formal language. But what kind of thing are extensions? Frege presented in *Gl* two candidates for the role: sets and aggregates. He had already spent eight paragraphs of his book (from §21 – 29) dealing with those two possible mathematical objects and their differences. That discussion was aimed at clearly distinguishing between these two candidates and favoring the construal of "extensions" as "sets". Our point here is that Frege did not introduce those logical entities, the extensions, in *Gl*. Let's see how he talked about them as they were used in the ordinary mathematician's practice then.

Frege first comment in §28 of *Gl* was that, although some writes take numbers to be "*Menge*", "multitude" or "plurality", which in German could also mean "set", their usage was not always very sharp. He argued that they could also have used other concepts to characterize what they meant, such as those of "heap", "group", or "aggregates". He concluded that their use of "Menge" was not precise and those other ideas were also present in their construction. In the following passage, Frege made a more clear-cut characterization of "Menge" dividing it into two groups: sets and aggregates.

> § 28. Some writers define Number as a *set* or *multitude* or *plurality*. All these views suffer from the drawback that the concept will not then cover the numbers 0 and 1. Moreover, these terms are utterly vague: sometimes they approximate in meaning to "heap" or "group" or "aggregate", referring to a juxtaposition in space, sometimes they are so used as to be practically equivalent to "Number", only vaguer. No analysis of the concept of Number, therefore, is to be found in a definition of this kind. (FREGE, The Foundations of Arithmetic, 1953)

In this passage, one of those terms – the word “sets” – seems to support Heck’s claim that *extensions* had always been *sets* for Frege even before Gg. However, based on other passages, we can suggest another interpretation. As we understand Frege, he was trying to distinguish those two kinds of entities: “sets” and “aggregates”. The distinction comes from his concern with another process of abstraction that was often confused with the one which originates the concept of “set”, the process of forming a concrete kind of entity by putting together some previously given physical bodies. This “aggregative” idea, criticized very often by Frege, lies subjacent to the widespread account of “multitude” or “plurality” frequently found in many mathematicians’ writings of his time.

In the sequence that follows the last quoted passage, Frege complains also about the vague use of the expression “concept extensions” by his fellows. His target was probably those who used it with the same meaning as “aggregates”. In *Gl,* Frege argued for the much more powerful idea of “falling under a concept” as a way to determine a set. In his famous example of the “*blind Germans*”, Frege praised the collecting power of concepts (1953, pp. 30, §23) and ridiculed the idea of aggregating or listing the elements of a set, one by one. His remark did not concern the extension themselves, nor the physical bodies which fall under them. Frege was concerned with the indispensability of the concept-words’ intensional feature, their sense (*Sinn*) in order to form a set of infinitely many elements. He seemed to be relying on the concepts’ “*intensional* power” of selecting the objects which belong to their extensions to define what he understood as an adequate construal of "sets".[18]

But, after criticizing so emphatically his colleagues, the next expected step for Frege should have been to propose his own definition of "extensions/sets", one that would be made from a strict and rigorous logical point of view. This, however, was not the path Frege took in *Gl*. He simply repeated some traditionally known definitions:

> The content of a concept diminishes as its extension increases; if its extension becomes all-embracing, its content must vanish altogether. (1953, pp. 40, §29)
>
> In this definition the sense of the expression "extension of a concept" is assumed to be known. (FREGE, The Foundations of Arithmetic, 1953, pp. 117, §107) [my emphasis]
>
> I assume that it is known what the extension of a concept is. (Note to §69)

In this passage, Frege was not giving a logical definition of "extension", nor was he trying to regiment the informal concept of "extension" that was erroneously used by his colleagues. Actually, he was doing quite the opposite. He was using the notion in the very same way as it was found by his fellows.

V.2 Footnote 1 of §69 revisited

Frege's suggestion of replacing concepts for its extensions in footnote 1 of §69 of *Gl* directly concerns all our debate up until now and is worth being dealt with once more here. In that famous footnote, reproduced in its integrity

[18] It is worth anticipating that the mistake of considering sets to be aggregates of physical bodies has been completely bypassed in Frege's construal of "extensions/value-ranges" in *Gg* with law V, as we are going to see next.

now, one could perhaps conclude that Frege was trying to give some kind of definition for the concept of “extension”. In fact, this passage is most frequently used to support this idea. What is argued is usually that Frege meant something in the following lines (for example, by Heck, above): *what I mean by “extensions of concepts” is equivalent to what I mean when I talk about the “concepts” themselves, or even vice-versa.*

> I believe that for “extension of the concept” we could write simply “concept”. But this would be open to the two objections: (1) that this contradicts my earlier statement that the individual numbers are objects, as is indicated by the use of the definite article in expressions like “the number two” and by the impossibility of speaking of ones, twos, etc. in the plural, as also by the fact that the number constitutes only an element in the predicate of a statement of number; (2) that concepts can have identical extensions without themselves coinciding. I am, as it happens, convinced that both these objections can be met; but to do this would take us too far afield for present purposes. (Note to §69)

Our interpretation of this passage is quite diverse from that though. We claim that Frege did not attempt to give a formal and complete definition of extension in this footnote. Given his subjunctive use of German verbs, we can raise the hypothesis that he was just doubtful about what to do.[19] Another argument in support of this hypothesis is that he raised two objections to his alleged definition, thus making it clear that he was still thinking about which candidate to adopt, “concept” or “extension of concept”. At the end of the footnote, Frege says that, although he had some ideas in mind

[19] “Ich glaube, dass für ‘Umfang des Begriffes’ einfach ‘Begriff’ gesagt **werden könnte**. Aber man würde zweierlei einwenden [...]” (Gl, note 87, §69).

about how to meet those objections, that was something he had not tried to do yet. We think that the correct way to read the footnote is to conclude that Frege was postponing the implementation of these ideas for another book. On our account, he had tried some of those ideas during the eight years of hard work and finally concluded that he needed both: concepts and their extensions. We think that in *Gl* Frege had not made up his mind as to how he should approach the subject yet.

Another commentary about objection (1) of footnote 1 is necessary. In *Gl*, Frege aimed to define numbers as objects and not as second level concepts. But if he had implemented the replacement of the word "extension of concept" with the word "concept", he would not be any closer to his aim. Quite on the contrary, Frege's definition would be saying that numbers are second-level properties, i.e., property of properties, and not objects. This was a big difficulty for Frege in *Gl*. According to his worries, if numbers were not "objects", they could not be objective entities, and neither would they be the adequate subjects of true mathematical propositions of identity.

Frege's next step must be to show how an identity proposition saying that two numbers are the same can be directly about numbers taken as objects. As numbers were first defined as second-level properties, Frege would have to show that, when two concepts were equal, then the numbers assigned to them should be equal too:

> the extension of the concept “equal to the concept F” is identical with the extension of the concept “equal to the concept G” is true *if, and only if,* the proposition “the same number belongs to the concept F as to the concept G” is also true. (1953, pp. 80, §69)

Right after presenting the definition above Frege started comparing numbers and extensions. According to him, one extension is wider or equal to another, but a number is greater or identical to the other: “Certainly, we do not say that one number is wider than another, in the sense in which the extension of one concept is wider than that of another; [...]” (1953, pp. 80, §69).

To cope with this asymmetric behavior, Frege proposed the following line of argumentation: two extensions cannot be equal (*gleichzahlig*) regarding a relation one-to-one if the numbers assignable to them are not and vice-versa.

In § 72 Frege reformulated his definition of §69, now enriched by the one-to-one relation of equinumerosity.

> The expression “the concept F is equal to the concept G” is to mean the same as the expression “there exists a relation Φ which correlates one to one the objects falling under the concept F with the objects falling under the concept G”. and add: the expression “*n* is a number” has the same meaning as the expression “there is a concept such that *n* is *the* number to which it belongs” (FREGE, The Foundations of Arithmetic, 1953, pp. 85, § 72)

In this passage, Frege tried to get closer to what he thought an identity proposition of number should look like. He finally added, in § 73 that: “the Number which belongs to the concept F is identical with the Number which belongs to the concept G if the concept F is equal to the concept G.”

In this passage, Frege's definitions of numbers are finally formulated as an identity. At each side of the verb "to be" we have the definite article followed by a property that identifies it. Only if the definition of "number" complies with this form a number could be correctly identified as an object in recognition judgments. Another noticeable movement taken by Frege in this last definition of §73 is to pass from a numerical identity between two numbers to an equality between concepts. As we have commented in section III.2 page 15, this was not a step Frege took without accompanying doubts. Those doubts concern the use of another sign for equality – a sign with three horizontal bars instead of just two – because what was really identifying those numbers were the two concepts and their equinumerousity. Besides, that last definition depended on the property of equinumerousity as much as the definition of direction depends on the property of parallelism, as we've discussed in section IV.

At §107, Frege was still in doubt about how to decide over recognition-judgements.

> The question then arose: when are we entitled to regard a content as that of a recognition-judgment? For this a certain condition has to be satisfied, namely that it must be possible in every judgment to substitute without loss of truth the right-hand side of our putative identity for its left-hand side. Now at the outset, and until we bring in further definitions, we do not know of any other assertion concerning either side of such an identity except the one, that they are identical. We had only to show, therefore, that the substitution is possible in an identity.

In this passage, unsatisfied with his interpretation given in §8 of *Bg* of identity propositions as definitions, and not judgments, Frege complained about Hume's principle as the only way to identify numbers in the domain.

> One doubt, however, still remained, which was this. A recognition-statement must always have a sense. But now if we treat the possibility of correlating one to one the objects falling under the concept P with the objects falling under the concept G as an identity, by putting for it: "the Number which belongs to the concept P is identical with the Number which belongs to the concept G", thus introducing the expression "the Number which belongs to the concept P", this gives us a sense for the identity only if both sides of it are of the form just mentioned. (1953, pp. 117, §107)

In § 108 Frege said that "It now still remained to define [the] one-one correlation; this we reduced to purely logical relationships." What seems to be missing then is a purely logical principle for obtaining the numbers as autonomous objects and so as possible arguments of any identity whatsoever. As we know, one of Frege's improvements in *Gg* and in his three middle papers was exactly this, i.e., to treat extensions as logical objects and so as the numerical reference of singular terms which describe them through their properties.

Looking closely at Frege's various complaints and summarizing our arguments, we could encounter a general pattern: the lack of universality in his first contextual definition of numbers identities. In *Gl*, his definition of "numbers" depended exclusively upon the special concept (relation) of equinumerosity. As we have already said, Frege's predicament was that numbers do reappear in other propositions but, in those occasions, we predicate about them different properties than that of equinumerosity. But we need also to be able to compare numbers with any other object in the domain, including physical bodies, as we have emphasized many times before. Proceeding now to the transitional period between G*l* and Gg, our next job will be to show how Frege completely changed his mind about the importance of "extensions of concepts" in his system during this period.

VI. The transitional period from *Gl* to *Gg*

VI.1 What has changed between *Gl* and *Gg*?

A lot had changed during the above-mentioned eight years delay of Frege's final masterpiece with respect to his semantics. In the second period of his work, Frege adopted a completely different method of analyzing the logical structure of propositions and a totally new way of dealing with his old construal of conceptual content, now divided between the two distinct notions of "sense" and "reference". Ontologically speaking, his new semantics gained an autonomous character and did not include the "context principle" anymore. It was much more akin to a compositional theory of reference, for only *after* fixing the referent and sense of a name, or the extension and sense of a concept, one should be able to obtain the reference and sense of any compound. Frege's new arrangement also respects a hierarchical organization. All functions would be disposed in their own level and would be applied to the level immediately below. In level zero, though, one should find only extensions, plus the object "Truth: and the object "False".

Besides distinguishing "senses" from "references", Frege has come to see the whole enterprise from a foundational and compositional point of view. Thus, the sense and the truth-value of any linguistic element should be derived respectively from the sense and reference of each participant of the universal domain of logical objects. It would then be in this universal domain that one should search for any criteria whatsoever for establishing the truth of identity propositions. Those propositions did not have a dual character anymore but should be taken uniformly as any other ordinary assertion.

At the time of *Gl*, Frege did not yet have his famous distinction between sense and reference. Also, he did not introduce a larger group of concepts that were used later on to build *Gg*'s reformed concept-script. So, the first thing to settle before our discussion of this transitional period can even begin is the list of what has changed in Frege's new conceptual framework. In the *Foreword* of *Gg*, Frege gives us a big help with this task, he provides us with a complete list of all the novelties and conceptual advances of his new book.

> This progress might be mentioned here briefly. The primitive signs used in my *Begriffsschrift* occur again here with one exception. Instead of the three parallel lines, I have chosen the usual equality-sign, for I have convinced myself that in arithmetic it possesses just that reference that I too want to designate. Thus, I use the word "equal" with the same reference as "coinciding with" or "identical with", and this is also how the equality-sign is actually used in arithmetic. [...]
>
> To the original primitive signs two have now been added: the smooth breathing, designating the value-range of a function, and a sign to play the role of the definite article in language.
>
> The introduction of value-ranges of functions is an essential step forward, thanks to which we achieve far greater flexibility. What previously had been derived signs can now be replaced by other, and indeed simpler, ones, although the definitions of single-validness of a relation, of following in a series, of mapping are essentially the same as those given partly in my *Begriffsschrift*, partly in my *Grundlagen der*

> *Arithmetik*. Value-ranges, however, have a much more fundamental importance; for I define *cardinal* numbers themselves as extensions of concepts, and extensions of concepts are value-ranges, according to my specification. So, without the latter one would never be able to get by. [my emphasis] (FREGE, Basic Laws of Arithmetic, 2013, p. IX)

In this long passage, one item stands out from all the others due to its crucial importance: the idea of "values-ranges" of functions. When Frege talked about his definition of cardinal numbers, he emphasized that "values-ranges" were of a "much more fundamental importance" than all the other novelties. Following Frege's opinion in this matter, we could say that his construal of this concept in Gg was completely different from anything he had ever done before. It was not the same concept he mentioned in G*l* and it had two new characteristics notes not yet attributed to it there.

Another change announced by Frege also stands out: the new equality sign. In the place of the three parallel lines used by him in the formalization of identity propositions in *Bg,* we now encounter just two parallel lines, the same we use in all ordinary equations of arithmetic. According to him, in the Gg's scenario, this sign is the correct one for expressing that two things are "the same", or two senses indicate the same object, or still that ***a*** "coincides with" or "is identical with" ***b***. We think that in Gg Frege finally becomes comfortable with the sign for objectual identity because his system was now able to include those identities as assertions about objects, and not about concepts, as was the case in G*l.* Those equations were not a consequence of the context principle and of the method of analysis jointly applied anymore.

Whenever it was necessary to compare two concepts of any level, law V could help indicate the appropriate identity between their extensions at level zero.

VI.2 Frege's uncertainty over the importance of the concept of "extension"

Since the beginning of his career, Frege has had this goal of explaining the objectual character of numbers in mathematics, as we have already said. Returning to *Gl*, in the next passage, numbers were characterized as the mathematical substance par excellence. That is to say, they should not be tributary to any other reality whatsoever.

> lt became clear that the number studied by arithmetic must be conceived not as a dependent attribute, but substantively. Number thus emerged as an object that can be recognized again, although not as a physical or even a merely spatial object. Nor yet as one of which we can form a picture by means of our imagination.
>
> [...] Now for every object there is one type of proposition which must have a sense, namely the recognition statement, which in the case of numbers is called an identity. (FREGE, The Foundations of Arithmetic, 1953, pp. 119, §106)

In this passage, Frege also identifies recognition-judgments about numbers as special cases of identity statements. In the following passage, from Gg, Frege repeats his conception that numbers must be objective and the subjects of recognition-judgments of identity.

> If there are logical objects at all – and the objects of arithmetic are such – then there must also be a means to grasp them, *to recognize them.* The basic law of logic which permits the transformation of the generality of an equality into an equality serves for

> this purpose. *Without such a means, a scientific foundation of arithmetic would be impossible*. For us it serves the purposes that other mathematicians intend to achieve by the creation of new numbers. [my emphasis] (FREGE, 2013, pp. 149, §147)

In this last passage, we can also attest to the relevance and importance for Frege of having a criterion for evaluating when we have two ways of presenting the same numbers through different properties. Right after mentioning those judgments, Frege says that the law which permits "the transformation of the generality of an equality into an equality" between objects, basic law V, is the instrument for passing from concepts to their extensions. According to him, without law V there would be no mathematical substance and it would be impossible to provide a solid ground for arithmetic.

> This resulted in our being unable to prove the identity of numbers. It became clear that the number studied by arithmetic must be conceived not as a dependent attribute, but substantivally. (FREGE, The Foundations of Arithmetic, 1953, pp. 116, §106)

The relevance of having the entire axiom and not just its lower part (something equivalent to *Hume's principle*) also becomes clear in the final part of the last passage. Contrary to Heck, we are confident that the upper part of Law V would be invaluable for introducing in the domain new extensions from already-known concepts. It would be essential whenever new numbers must be included into the domain. Without law V, this introduction process would be an arbitrary procedure, without any objective justification.

The step from functions to their extensions was also a crucial and essential one when it comes to Frege's account of the reals. The extraordinary emphasis given by him to his new logical law is even more astonishing if we compare it with the later position held by him in *Gl.* In this previous work, the lack of importance given by him to the concept of "extension" was, indeed, quite evident:

> The Number which belongs to the concept P is the extension of the concept "concept equal to the concept p", where a concept P is called equal to a concept G if there exists the possibility of one-one correlation referred to above. [...] This way of getting over the difficulty cannot be expected to meet with universal approval, and many will prefer other methods of removing the doubt in question. *I attach no decisive importance even to bringing in the extensions of concepts at all.* (FREGE, The Foundations of Arithmetic, 1953, pp. 117, §107) [my emphasis]

Comparing this passage with the one quoted at the beginning of this section, the following conclusion becomes inevitable. What in *Gl* was one possible criterion among other alternative solutions for "removing the doubt in question", in *Gg* became the unique alternative for justifying any identity, including the arithmetical ones, one "without [which], a scientific foundation of arithmetic would be impossible". But why did Frege make such a radical change in his priorities? In *Gl*, we could even risk saying that "the construal of 'the extension of a concept' was an almost lateral strategy". In *Gg*, however, it became the sole way to solve all his difficulties. Of course, such a substantial change would not be a decision undertaken smoothly by Frege.

We think that Frege was very hesitant about accepting that all concepts could have an extensional counterpart representing them in the domain. In fact, the introduction of "new objects" into the domain was the non-logical (ontological or semantical) consequence he feared most. We guess that it was this high ontological cost that discouraged him in the beginning. For him, furthermore, Law V did not have the same transparent character, the same clarity (*einleuchtend)* as the other logical laws. It was not a truism and it did not seem to be self-standing without the need for further validation proofs either. Frege's apprehension was that this law might turn out not to be a logical law after all, as he himself later explains.

> As far as I can see, a dispute can arise only concerning my basic law of value-ranges (V), which perhaps has not yet been explicitly formulated by logicians although one thinks in accordance with it if, e.g., one speaks of extensions of concepts. I take it to be purely logical. At any rate, the place is hereby marked where there has to be a decision.

After the paradox, his reaction was:

> This is the position into which I was put by a letter from Mr. Bertrand Russell as the printing of this volume was nearing completion. The matter concerns my basic law (V). I have never concealed from myself that it is not as obvious (*einleuchtend*) as the others nor as obvious[20] as must properly be required of a logical law. Indeed, I pointed out this very weakness in the foreword to the first volume, p. VII. *I would gladly have dispensed with this foundation if I had known of some substitute for it.* Even now, I do not see how arithmetic can be founded scientifically, how the numbers can be apprehended as logical objects and brought under

[20] The word "einleuchtend" was translated by Philip A. Ebert & Marcus Rossberg WITH Crispin Wright, Heck and his Fellows by "obvious", but an extensive commentary can be found about it in (FREGE, Basic Laws of Arithmetic, 2013, p. xxii).

> consideration, if it is not – at least conditionally – permissible to pass from a concept to its extension. [my emphasis] (FREGE, Basic Laws of Arithmetic, 2013, pp. 253, vII, appendix)

In this passage, the transition from concepts to their extensions was repeatedly emphasized as a crucial step for Frege. He was explicit about his difficulties in seeing another way to include all numbers in the universal domain. It can also be apprehended from this passage that Frege had tried some other methods for achieving this same result without success.[21] Assuming then that this was effectively what happened during those eight years, our answer to Heck's question should be reformulated as follows: Frege's "logical law V" was the only way to overcome his earlier frustrated attempts to find a universal criterion for identity propositions of number and so to treat numbers as the mathematical substance *par excellence*. Probably at the end of the period, Frege was simply forced to stipulate law V despite all the risks involved.

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[21] As we said earlier, in *Gl* Frege had tried to associate two philosophical principles with the objective of providing for the conceptual content of all expressions including the names of number: (1) the fertility of his analytical method; plus (2) the context principle. As we have concluded before, that strategy has proved to be insufficient for providing identity criteria for all recognition-judgments of numbers. We can risk being even more insistent: Frege had probably tried variations of them, as well as a weakened version of law V like Hume's principle, for example, during those eight years of trials and finally comes to the conclusion that none of those alternatives could ever support the demands made upon them by the recognition-judgments. Frege definition of Hume's principle was: "HUME long ago mentioned such a means: "When two numbers are so combined as that the one has always a unit answering to every unit of the other, we pronounce them equal". (FREGE, The Foundations of Arithmetic, 1953, p. 73. §63)

VII. Extensions as a criterion for recognition judgments in *Gg*.

After a long period of struggle, Frege finally gave up the search for other methods and decided to assume extensions as the only way out of his difficulties. To keep the universality of identity and include the numbers as subjects of identity statements, Frege stipulated in Gg that extensions were logical objects and the only representatives of concepts at level zero.

In the 10th first paragraphs of Gg Frege introduced his way to present the basic laws of his system: by what he called "stipulations" (*Festsetzen*).[22] Stipulations for Frege were laws of a logical nature. They do not need to be deduced from other more basic laws and they do not need any further proof of adequacy besides their logical character of truisms. In Frege's words:

> The ambiguity of the word "law" here is fatal. ln one sense it says what is, in the other it prescribes what ought to be. Only in the latter sense can the logical laws be called laws of thought, in so far as they legislate how one ought to think. (FREGE, Basic Laws of Arithmetic, 2013, p. xv)

[22] We quote the translators" explanation for favoring the word "stipulate" and not "define", "fix" or "legislate". "There are two exceptions to our translation of "Festsetzung" and "festsetzen" as "stipulation" and "stipulate". ln the foreword, p. XV, we use "legislate" to reflect a legal connotation of "festsetzen". ln this passage, Frege is discussing the normative aspect of laws of thought and the extent to which they "legislate" ("festsetzen") how one ought to think. Given that laws do not, strictly speaking, "stipulate" anything in the sense in which "stipulate" is used elsewhere in Grundgesetze, Frege's use of "festsetzen" here is better captured by "legislate". ln vol. II, p. 94, §83, we use "fix" instead of "stipulate" as a translation of "festsetzen": "These definitions can fix the references of the new words and signs with at least the same right as [...]". Using "stipulate" here could give the impression that Frege intends to stipulate objects into existence, while the German original clearly does not invite this reading." (FREGE, Basic Laws of Arithmetic, 2013, p. xx)

This passage is close to another one that opens his later paper "*Der Gedanken*". Here, as well as there, Frege was concerned with distinguishing mathematics and logic from psychology. Frege had always wanted to prove that mathematical propositions were "about something" and not "about nothing". He wanted to show that they did have autonomy from our way of thinking and were not like laws of psychological science which describe the way we humans actually process our thoughts. For him, mathematical propositions' truth must be grounded in an objective reality. The ultimate ground for Frege were logical laws, for they could function as principles from which all mathematical truth could be derivate. But Frege considered logical laws as akin to laws of ethics and not as laws of science. The former, but not the latter, established the parameters for correct thinking. Finally, when stating logical laws, one must be very careful, as they could give rise to "artificial objects" that would be introduced into the domain only for theoretical needs. In the following passage, Frege was worried about just this issue.

> How much easier this would be if one could simply create the required object! If we do not know whether there is a number whose square is -1, then we simply create one. If we do not know whether some prime number has a primitive root, then we simply create one. [...] This, unfortunately, is too convenient to be correct. Certain constraints on creating have to be acknowledged. For an arithmetician who accepts the possibility of creation in general, the most important thing will be a lucid development of the laws that are to govern it, in order then to prove of each individual act of creation that it is sanctioned by these laws. Otherwise, everything will be imprecise, and the proofs will descend to a mere illusion, to a gratifying self-deception. (2013, pp. 142, §140, vol II, part III)

In this passage, Frege strongly rejected a common way to solve problems in mathematics, the creation of new numbers. In part III, of volume II, Frege compared his new way of converting "the generality of an equality [...] into an equality (identity)" generating an extension through axiom V with what those mathematicians who create new numbers at will do when stating their systems:

> [here] the generality of an equality is [...] converted into an equality (identity). When logicians have long spoken of the extension of a concept and mathematicians have spoken of sets, classes, and manifolds, then such a conversion forms the basis of this too; for, one may well take it that what mathematicians call a set, etc., is really nothing but the extension of a concept, even if they are not always clearly aware of this.
>
> We are thus not really doing anything new by means of this conversion; but we do it in full awareness and by appealing to a basic law of logic. And what we do in this way is completely different from the arbitrary, lawless creation of numbers by many mathematicians. (2013, pp. 148, §147, vol II, Part III)

According to Frege in this passage, we are not really doing anything new by means of this conversion; besides, we do it in full consciousness and by appealing to a basic law of logic. And thus, for him, what he did with law V was completely different from the arbitrary, lawless creation of numbers made by the other mathematicians.

Frege thought about his stipulations as having a very different character from the arbitrary assumptions of his colleagues. In a footnote, Frege gives the following explanation: "In general, we should not regard the stipulations about the primitive signs in the first volume as definitions. Only what is logically composite can be defined; what is simple can only be pointed to." (2013, pp. 148, §147, vol II, Part III, ft 1)

According to him, *Basic Law V*, presented in the 11th paragraph of Gg, was a stipulation[23]. For him, it had a normative character, for it had to do with how one ought to think and not with what went on in reality. In the case of law V, it states that extensions, being the representatives of properties at level zero, *are logical objects* and, actually, *the only objects to be considered in Frege's formal language besides the Truth and the False.*

Frege introduced Law V by steps in Gg, though. First, in paragraph 9, he stipulates that every intensional property corresponds to an extension:

> If $\overset{\mathfrak{a}}{\smile}\ \Phi(a) = \Psi(a)$ is the True, we can, according to our previous specification (§3), also say that the function $\Phi(\xi)$ has the same value-range as the function $\Psi(\xi)$; that is: we can convert the generality of an equality into a value-range equality and vice versa. This possibility must be regarded as a logical law of which, incidentally, use has always been made, even if tacitly, whenever extensions of concepts were mentioned. The entire calculating logic of Leibniz and Boole rests upon it. (FREGE, Basic Laws of Arithmetic, 2013, p. 14).

Then he adds to this logical law another stipulation which ruled out the case of objects that were not introduced as extensions in the universal domain:

[23] We think that the American translators were right in being preoccupied with their translation of the words "Festsetzung" and "festsetzen" as "stipulation" and "stipulate". According to them (as we transcribe in note 11), the German word has also the sense of "define", "fix" or "legislate", and stipulate" could give the impression that Frege "intends to stipulate objects into existence, while the German word clearly does not invite this reading.". We decided to use their translation with this proviso.

> It suggests itself to generalize our stipulation so that every object is conceived as a value-range, namely, as the extension of a concept under which it falls as the only object. A concept under which only the object Δ falls is $\Delta = \xi$. We attempt the stipulation: let $\acute{\varepsilon}(\Delta = \varepsilon)$ be the same as Δ. Such a stipulation is possible for every object that is given to us independently of value-ranges [...] (FREGE, Basic Laws of Arithmetic, 2013, pp. 18, footnote 1).

Together, they were responsible for grounding Frege's compositional language on a single universal domain of logical objects.

One caution must be observed though. The new logical objects included in the domain were mere "representatives of properties". Frege was explicit about this point. He emphasized that extensions could not "replace" their functions. Even when they were used as the argument of other higher-level functions, their roles would only be of "*representatives*" of their respective functions. For him, they would serve just as an "economical device" to express functions of any level as arguments of other functions without creating a special notation for that.

> §25. [...] Here it may merely be briefly remarked that this economy is made possible by the fact that second-level functions are representable, in a way, by first-level functions where the functions that appear as arguments of the former are represented by their value-ranges. (FREGE, Basic Laws of Arithmetic, 2013, p. §25).
>
> As was indicated, [...] functions appearing as arguments of second-level functions are represented by their value-ranges, although of course not in such a way that they simply concede their places to them, *for that is impossible*. [my emphasis] (FREGE, Basic Laws of Arithmetic, 2013, p. §34)

In Gg's Foreword, Frege had advised us already about the correct interpretation of his stipulations in Gg:

> When one has reached the end, one should reread the entire exposition of the concept-script with this as background, keeping in mind that those *stipulations* that will not be used later, and therefore appear unnecessary, serve to implement the principle that all correctly formed signs ought to refer to something – a principle that is essential for full rigor. (FREGE, Basic Laws of Arithmetic, 2013, p. XII)

In this passage, Frege clearly emphasizes the essentiality of the top intensional part of his law V in the system.

VIII. Conclusive remarks about Frege's Law V and the Real numbers

We want to add the last remark about logical law V. Frege's extensional universal domain was a key element for the complete determination of the sense of any linguistic expression incorporated in his system. At the time of Gg, he thought that his new conception of "extension" would do the job. So, all deductive chains which introduced new concepts and were used to form new singular terms would include new logical objects, the extensions, in the domain. Besides, any new concept would inaugurate for Frege a new context concerning the application of the identity sign. For him, every new concept must be submitted to new tests involving identity. This means that one should be able to determine if their extensions, or the objects named by singular terms which involve them, are one and the same thing. In the case of not having something like Frege's law V, one would have to solve the following problem: every new concept and every new object would demand a reformulation of the identity sign.

In part III of Gg, where Frege tried to present what was to be the crowning of his efforts, an explanation of the mathematical nature of Real numbers, he accused once again other mathematicians of being somewhat sloppy. Their fault was always the same: to introduce new mathematical objects slovenly, neglectfully of how those new objects will affect the entire system when used flanking the identity sign. He criticizes them heavily in this and other passages, as we have seen him doing so many times, repeatedly emphasizing the only path he thought would be worth following:

> Often, both of our principles of definition are flouted at once, for example, by explaining the equality-sign together with what stands on its right and left. [...] On the one hand, it seems one is supposed to recall the earlier definition and elicit from it something to determine what now occurs to the right and left. But on the other, this earlier explanation does not suffice for the case at hand. Something similar also happens with other signs. This twilight is required by some mathematicians for the performance of their logical conjuring tricks. *The results that are to be gained in this way may be obtained in an irreproachable manner by our conversion of the generality of an equality into an equality of value-ranges in accordance with basic law V* (vol. I, §3, §9, §20) [my emphasis]

Frege does sound very confident in this passage. After introducing his principles and being sure about the importance of the new identity criteria provided by his fifth axiom, he appears to foresee the worst-case scenario: a disaster that would follow from never being able to obtain an axiom like his.

> If mathematicians' opinions about equality diverge, then this means nothing less than that mathematicians disagree with respect to the content of their science; and if one regards the essence of the science as being thoughts, rather than words or signs, then this means that there is no one united mathematical science, that mathematicians do not, in fact, understand each other. (FREGE, Basic Laws of Arithmetic, 2013, pp. 71, §58, nota 1)

Frege's attitude of completely giving up everything in face of Russell's paradox was more than understandable if we look at the situation in the same way as him. It is even more comprehensible if we pay attention to the consequences of Gödel's first incompleteness theorem, produced by the option of reinterpreting numerals, sometimes as numbers, sometimes as names of syntactic sequences. Frege seems also to fear something along these lines: "It is an atrocious state of affairs that the use of the word "number" by mathematicians fluctuates sometimes it is the number-signs that are called numbers, sometimes it is their reference." (FREGE, Basic Laws of Arithmetic, 2013, pp. 80, §68)

One final argument supporting our claim occurs in an epistolary exchange between Frege and Russell. It began with Russell's realization of the importance and centrality of Frege's concept of "extension" interpreted as a "logical object":

> Many thanks for your explanations concerning value-ranges. I now understand the necessity of treating ranges of values not just as aggregates of objects or as systems. But I still lack a direct intuition, a direct insight into what you call a range of values: logically it is necessary, but it remains for me a justified hypothesis. (Russell's letter to Frege, 1902). (FREGE, 1980, p. 143-144).

Russell ended his answer by asking for more explanations about the status of this central concept in Frege's system. The lack of a proper understanding however caused Russell to urge Frege into trying to find a solution that would make the consequences of the paradox disappear. Russell proposed that Frege could face the setbacks presented in his letter by incorporating a subterfuge into his system, a partial definition of identity. In other words, Russell's suggestion was to treat extensions in two separate ways, sometimes as "proper" objects, and sometimes as "improper" ones. However, from all that we have discussed in this paper, including Frege's arguments in favor of one single concept of "identity" for the entire language, his reply should come as no surprise.

> If we wanted to revoke the law of excluded middle for classes, we could consider taking classes – and presumably value-ranges in general – as improper objects. ln that case, they would not be admissible as arguments for all first-level functions. There would, however, be some functions which could have both proper and improper objects as arguments. At least the relation of equality (identity) would be of this kind. One might try to avoid this by assuming a special kind of equality for improper objects. But that is surely ruled out. (FREGE, Basic Laws of Arithmetic, 2013, pp. 254, Afterword)

In this passage, Frege presents his strongest argument against any alternative solution to the recognition-judgments' problem. For him, Russell's solution would involve revoking the excluded middle for classes/value-ranges in general.[24] Frege's idea of a universal domain go logical objects

[24] According to Frege's formulation of this principle provided in the first two paragraphs of part III of Gg: "The law of excluded middle is in fact just the requirement, in another form, that concepts have sharp boundaries. Any object Δ either falls under the concept *P*, or it does not fall under it: *tertium non datur*." (FREGE, Basic Laws of Arithmetic, 2013, pp. 70, §56, part III)

was of course one of the roots of those difficulties. But the other one was that all concepts should be completely determined, and this requirement surely includes first and above all the concept of "identity". We think that Frege's certainty came from his deep convictions that one's primary concern should be with keeping the universal validity of the *tertium non datur* for all concepts – without having a piecemeal treatment of identity. We conclude our paper with Frege's own words: "Identity is a relation given in so determinate a way that it is inconceivable that different kinds of it could occur. But now the result would be a great multitude of first-level functions, [...]" (FREGE, Basic Laws of Arithmetic, 2013, pp. 254, part III)

Resumo: O objetivo deste artigo é responder a uma questão proposta por Richard Heck no artigo "Formal Arithmetic Before Grundgesetze". Nesse artigo, Heck indaga a respeito das razões pelas quais Frege levou quase oito anos para honrar suas promessas de concluir seu grandioso projeto de fundamentar a matemática na lógica. Embora Heck tenha tentado responder a sua própria pergunta, pensamos que uma discussão filosófica mais adequada sobre o atraso de Frege ainda pode ser oferecida. Este artigo tentará preencher essa lacuna apresentando o que entendemos ser o problema central enfrentado por Frege em *Die Grundlagen der Arithmetik*: a falta de um critério unificado para fixar o significado das proposições de identidade da matemática. Acreditamos que a proposta inicial de Frege de atribuir um duplo papel para "proposições de identidade" foi a causa de todos os seus problemas em fixar sua definição de números em *Die Grundlagen der Arithmetik*. No período de oito anos mencionado, o desafio de Frege passou a ser o de encontrar um critério capaz de unificar o seu tratamento das identidades. Acreditamos que o filósofo alemão finalmente tenha decidido preencher essa lacuna, fornecendo uma nova interpretação para o conceito de "extensão", uma que acrescentasse alguns refinamentos importantes a sua concepção anterior. O novo conceito assim construído permitiu a Frege unificar seu tratamento das proposições de identidade, incluindo em seu sistema um critério universal e flexível para decidir a verdade de qualquer proposição de identidade. A nova interpretação do conceito de "extensão" de Frege foi apoiada por sua famosa lei básica V. Assim, nossa alegação será que a resistência de Frege e as dúvidas sobre a inclusão do axioma V como uma lei lógica em seu sistema foram a causa primária desse atraso.

Palavras-chave: Frege, extensão, identidade.

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