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Wrapped Birnbaum-Saunders distribution: definition and estimation

Distribuição Wrapped Birnbaum-Saunders: Definição e Estimação

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Abstract:

In this work a new circular distribution called wrapped Birnbaum-Saunders was proposed. It was obtained by wrapping the classical Birnbaum-Saunders distribution in a reparameterized form. For this distribution we have found expressions for their probability density function, distribution function and trigonometric moments. We show some properties of this new distribution and obtained the maximum likelihood estimators of its two parameters, in addition, we conducted a Monte Carlo simulation study to evaluate the performance of the maximum likelihood estimators of the parameters. We also made an application to a real dataset from the Rudolf Jander's experiments concerning the direction chosen by ants in response to a stimulus and compare its estimates via Kuiper's statistic with those obtained from the Von Mises and Asymmetric Von Mises models. This distribution is very promising as model for asymmetric directional data.

Keywords: Circular data. Wrapped Birnbaum-Saunders distribution. Birnbaum-Saunders distribution. Maximum likelihood.

Resumo:

Neste trabalho propomos uma nova distribuição circular chamada de wrapped Birnbaum-Saunder. Essa distribuição foi obtida por meio do arqueamento da distribuição Birnbaum-Saunders clássica em sua forma reparametrizada. Para esta distribuição, nós encontramos as expressões para sua função densidade de probabilidade, função de distribuição e momentos trigonométricos. Nós mostramos algumas propriedades desta nova distribuição e obtemos os estimadores de máxima verossimilhança para seus dois parâmetros, além disso, conduzimos um estudo de simulação de Monte Carlo para avaliar a performance destes estimadores

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dos parâmetros. Também fizemos uma aplicação a um conjunto de dados reais obtidos do experimento de Rudolf Jander a respeito da direção escolhida por formigas em resposta a um estímulo e comparamos as estimativas obtidas, por meio da estatística de Kuiper, com as estimativas obtidas pelos modelos Von Mises e Von Mises assimétrica. Esta distribuição é bastante promissora como um modelo para dados circulares assimétricos.

Palavras-chave: Dados circulares. Distribuição Wrapped Birnbaum-Saunders . Distribuição Birnbaum-Saunders . Máxima verossimilhança.

1 Introduction

In this work we present the wrapped version of the Birnbaum-Saunders distribution. The classical Birnbaum-Saunders distribution was present by [1] for fatigue live of structures under cyclic stress. A reparameterized version was proposed by [16] in terms of the parameters of the classical Birnbaum-Saunders distribution. [17] provides some results on moments, estimation and generation method of random numbers from this reparameterized version of the Birnbaum-Saunders distribution.

Circular data arise naturally in many fields of science, such as medicine, biology, physics, meteorology, among others [10] and, in general, the circular measurements associated with these data are recorded in directional or periodic phenomena, such as the direction of movement of an animal after a stimulus (directional) or the time of arrival of a patient in a hospital emergency room (periodic). Circular distributions are used to model circular data and these can be characterized by being symmetrical or asymmetrical, among other features of interest. Even though most circular distributions are symmetrical, there are several practical situations where skewed distributions are necessary and, in general, it is difficult to deal with skewed models and this difficulty, in part, is due to the lack of some mathematical properties that symmetrical models have. For example, asymmetric models often have complex normalizing constants (complex number) and also complex trigonometric moments, which can cause problems in the analyses.

Levy [9] proposed a method to introduce a wrapped variable in a symmetric or asymmetric distribution, which is known in the literature as the wrapped distribution and since then several works have been published in this area. Wrapped distributions constitute a very rich and useful class of probabilistic models for circular data. Jammalamadaka and Kozubowski [5] discussed circular distributions obtained through the wrapped method from exponential and Laplace distributions. Coelho [2] obtained an expression for the probability density function of the wrapped gamma distribution, obtained from the traditional gamma distribution, and showed how it can be obtained as a mixture of truncated gamma distributions. Rao, Sarma and Girija [13] derived new circular models from the already known lifetime distributions, such as lognormal, logistic, Weibull and the distribution of extreme values. Roy and Adnan [15] explored the wrapped Gompertz distribution and discussed its application in ornithology. In another work, Roy and Adnan [14] developed a new class of circular distributions called wrapped exponential weighted. Jacob and Jayakumar [4] proposed a new family of circular distributions through geometric wrapped and studied their properties. Rao, Girija and Devaraaj [12] discussed the characteristics of the wrapped gamma distribution. Joshi and K [8] proposed a wrapped version of the Lindley distribution and derived expressions for its characteristic function, trigonometric moments, skewness coefficients and kurtosis.

There are difficulties in obtaining circular models with tractable asymmetry and trigonometric moments, therefore, to try to overcome these difficulties, our work came to fill this gap in the literature, in which a new wrapped distribution is proposed from the reparameterized Birnbaum-Saunders [16]. We present its probability density function, cumulative distribution function and trigonometric moments, in addition, we carry out a Monte Carlo simulation study with the objective of evaluating the performance of the maximum likelihood estimators of the model parameters and we also made an application to a real data set.

This paper is organized as follows. The Section 2 presents the Wrapped Birnbaum-Saunders distribution and its properties. Section 3 shows the its trigonometric moments. The Section 4 presents the score functions and shows that the maximum likelihood estimators has no closed form for the Wrapped Birnbaum-Saunders distribution parameters. Section 5 describe the simulation and show the obtained results. The Section 6 presents the application to a real data set. Finally, Section 7 presents our conclusions.

2 The Wrapped Birnbaum-Saunders Distribution

Remark 1. *Let Y be a real random variable with probability density function f, cumulative distribution function F and characteristic function φ. Then the corresponding wrapped random variable is given by*

$$\theta = Y \pmod{2\pi}$$

and has the probability density function, cumulative distribution function and characteristic function given, respectively, by

$$
\begin{aligned}
f_\theta(\theta) &= \sum_{k=-\infty}^{k=+\infty} f(\theta + 2k\pi),\ \theta \in [0, 2\pi),\\
F_\theta(\theta) &= \sum_{k=-\infty}^{k=+\infty} \left\{F(\theta + 2k\pi) - F(2k\pi)\right\},\ \theta \in [0, 2\pi),\\
\varphi_p &= \varphi(p) = \mathbb{E}\left[\mathrm{e}^{ip\theta}\right],\ p \in \mathbb{Z}.
\end{aligned}
$$

See [3], [6] and [10] for details.

Birnbaum and Saunders [1] introduced his classical distribution indexed by two parameters where α is the shape parameter, whereas β is the scale parameter. [17] shows a new parameterization of the Birnbaum-Saunders distribution indexed by the parameters μ and δ, where $0 < \mu = \beta\,(1 + \alpha^2/2)$ is a scale parameter and the mean of the distribution, whereas $0 < \delta = 2/\alpha^2$ is a shape and precision parameter. Based on this parameterization we have the follow.

Remark 2. *Let Y a random variable with the probability density function given by*

$$
f_Y(y;\mu,\delta) = \frac{\mathrm{e}^{\delta/2}\sqrt{\delta+1}}{4\sqrt{\mu\pi}y^{3/2}}\left[y + \frac{\delta\mu}{\delta+1}\right]\exp\left\{-\frac{\delta}{4}\left[\frac{y(\delta+1)}{\delta\mu} + \frac{\delta\mu}{y(\delta+1)}\right]\right\},
$$

where $y > 0$, $\mu > 0$ and $\delta > 0$. Then Y is a Birnbaum-Saunders distribution indexed by the parameters μ and δ. The notation $Y \sim \mathcal{BS}(\mu,\delta)$ is used.

[17] shows that $\mathbb{E}[Y] = \mu$ and $\mathrm{Var}[Y] = \frac{\mu^2(2\delta+5)}{(\delta+1)^2}$.

An important property of the Birnbaum-Saunders distribution is its relashionship with the normal distribution. This relationship is given as follow ([17]). If $Y \sim \mathcal{BS}(\mu,\delta)$ and $Z \sim \mathrm{N}(0,1)$, then

$$
Y = \frac{\delta\mu}{\delta+1}\left[\frac{Z}{\sqrt{2\delta}} + \sqrt{\left\{\frac{Z}{\sqrt{2\delta}}\right\}^2 + 1}\right]^2 \sim \mathcal{BS}(\mu,\delta)
$$

and

$$
Z = \sqrt{\frac{\delta}{2}}\left[\sqrt{\frac{(\delta+1)Y}{\delta\mu}} - \sqrt{\frac{\delta\mu}{(\delta+1)Y}}\right] \sim \mathrm{N}(0,1). \tag{2.1}
$$

The corresponding wrapped random variable of Y is given as follow.

Definition 1. *Let $Y \sim \mathcal{BS}(\mu,\delta)$ then $\theta \equiv Y(\mathrm{mod}\ 2\pi)$ follows a Wrapped Birnbaum-Saunders distribution indexed by the parameters μ and δ with probability density* *function given by*

$$
f_\theta(\theta) = \sum_{k=0}^{\infty} f_Y(\theta + 2k\pi) = \tag{2.2}
$$

$$
\frac{e^{\frac{\delta}{2}}\sqrt{\delta+1}}{4\sqrt{\pi\mu}} \sum_{k=0}^{\infty} \frac{\theta + 2k\pi + \frac{\delta\mu}{\delta+1}}{(\theta+2k\pi)^{3/2}} \exp\left\{-\frac{\delta}{4}\left[\frac{(\theta+2k\pi)(\delta+1)}{\delta\mu} + \frac{\delta\mu}{(\theta+2k\pi)(\delta+1)}\right]\right\},
$$

where $\theta \in [0, 2\pi)$, $\mu \in [0, 2\pi)$ is the circular mean of the distribution and $\delta > 0$ is the shape parameter. The notation $\theta \sim \mathcal{WBS}(\mu, \delta)$ is used.

Proposição 1. *The series*

$$
\frac{e^{\delta/2}\sqrt{\delta+1}}{4\sqrt{\pi\mu}} \sum_{k=0}^{\infty} \frac{\theta + 2k\pi + \frac{\delta\mu}{\delta+1}}{(\theta+2k\pi)^{3/2}} \exp\left\{-\frac{\delta}{4}\left[\frac{(\theta+2k\pi)(\delta+1)}{\delta\mu} + \frac{\delta\mu}{(\theta+2k\pi)(\delta+1)}\right]\right\} \tag{2.3}
$$

given in 2.2 is a convergente series.

Proof. By the d'Alembert's ratio test. The geral term of the series is

$$
a_n = \frac{\theta + 2n\pi + \frac{\delta\mu}{\delta+1}}{(\theta+2n\pi)^{3/2}} \exp\left\{-\frac{\delta}{4}\left[\frac{(\theta+2n\pi)(\delta+1)}{\delta\mu} + \frac{\delta\mu}{(\theta+2n\pi)(\delta+1)}\right]\right\}.
$$

We have that $a_n > 0$ because $\theta, \mu, \delta > 0$. So

$$
\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\frac{\theta+2(n+1)\pi+\frac{\delta\mu}{\delta+1}}{(\theta+2(n+1)\pi)^{3/2}} \exp\left\{-\frac{\delta}{4}\left[\frac{(\theta+2(n+1)\pi)(\delta+1)}{\delta\mu} + \frac{\delta\mu}{(\theta+2(n+1)\pi)(\delta+1)}\right]\right\}}{\frac{\theta+2n\pi+\frac{\delta\mu}{\delta+1}}{(\theta+2n\pi)^{3/2}} \exp\left\{-\frac{\delta}{4}\left[\frac{(\theta+2n\pi)(\delta+1)}{\delta\mu} + \frac{\delta\mu}{(\theta+2n\pi)(\delta+1)}\right]\right\}} \\
\frac{a_{n+1}}{a_n} &= \left[1 + \frac{2\pi}{\theta + 2n\pi + \frac{\delta\mu}{\delta+1}}\right]\left[1 + \frac{2\pi}{\theta+2n\pi}\right]^{-\frac{3}{2}} \\
&\quad \times \exp\left\{-\frac{\delta}{4}\left[\frac{2\pi(\delta+1)}{\delta\mu} + \frac{2\pi\delta\mu}{(\theta+2n\pi)(\theta+2(n+1)\pi)(\delta+1)^2}\right]\right\}.
\end{aligned}
$$

Now note that

$$\begin{aligned}
&\lim_{n\to\infty} 1 + \frac{2\pi}{\theta + 2n\pi + \frac{\delta\mu}{\delta+1}} = 1\\
&\lim_{n\to\infty} 1 + \frac{2\pi}{\theta + 2n\pi} = 1\\
&\lim_{n\to\infty} \exp\left\{-\frac{\delta}{4}\left[\frac{2\pi(\delta+1)}{\delta\mu} + \frac{2\pi\delta\mu}{(\theta+2n\pi)(\theta+2(n+1)\pi)(\delta+1)^2}\right]\right\}\\
&\qquad = \exp\left\{-\frac{\delta}{4}\left[\frac{2\pi(\delta+1)}{\delta\mu}\right]\right\} = \exp\left\{-\frac{1}{2}\left[\frac{\pi(\delta+1)}{\mu}\right]\right\}.
\end{aligned}$$

Then $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \exp\left\{-\frac{1}{2}\left[\frac{\pi(\delta+1)}{\mu}\right]\right\} < 1$ and therefore the series converges absolutely. □

In practice, the number of parcels in the summation is fixed or we have to determine some stopping criterion. We have determined $k = 8$ that insure a very good approximation.

Proposição 2. *Let $Y \sim \mathcal{BS}(\mu,\delta)$ and let θ be its corresponding wrapped random variable, that is, $\theta \sim \mathcal{WBS}(\mu,\delta)$. Then*

$$\mathbb{E}[\cos\theta] \simeq \cos(\mu)\left[1 - \tfrac{\mu^2(2\delta+5)}{2(\delta+1)^2}\right] \textit{ and } \mathbb{E}[\sin\theta] \simeq \sin(\mu)\left[1 - \tfrac{\mu^2(2\delta+5)}{2(\delta+1)^2}\right].$$

Proof. We know that $\theta \equiv Y (\mathrm{mod}\ 2\pi)$ if, and only if, $\exists \kappa \in \mathbb{Z}$ so that $\theta = Y + 2\kappa\pi$. Then

$$\cos(\theta) = \cos(Y + 2\kappa\pi) = \cos(Y)\cos(2\kappa\pi) - \sin(Y)\sin(2\kappa\pi) = \cos(Y).$$

Similarly,

$$\sin(\theta) = \sin(Y + 2\kappa\pi) = \sin(Y)\cos(2\kappa\pi) + \cos(Y)\sin(2\kappa\pi) = \sin(Y).$$

Then we have $\cos(\theta) = \cos(Y)$ and $\sin(\theta) = \sin(Y)$, follow that $\mathbb{E}[\cos(\theta)] = \mathbb{E}[\cos(Y)]$ and $\mathbb{E}[\sin(\theta)] = \mathbb{E}[\sin(Y)]$. By the Taylor's expansion of second order of $\cos(Y)$ and of $\sin(Y)$ around μ (circular mean), we have the follow:

$$\begin{aligned}
\mathbb{E}[\cos(Y)] &\simeq \cos(\mu) + \frac{1}{2}\cos''(\mu)\mathbb{E}(Y-\mu)^2 = \cos(\mu) + \frac{1}{2}\cos''(\mu)\mathrm{Var}(Y),\\
\mathbb{E}[\sin(Y)] &\simeq \sin(\mu) + \frac{1}{2}\sin''(\mu)\mathbb{E}(Y-\mu)^2 = \sin(\mu) + \frac{1}{2}\sin''(\mu)\mathrm{Var}(Y)
\end{aligned}$$

where $Y \sim \mathcal{BS}(\mu, \delta)$. According to [17] $\mathrm{Var}(Y) = \frac{\mu^2(2\delta+5)}{(\delta+1)^2}$, therefore

$$\begin{aligned}\mathbb{E}[\cos(\theta)] = \mathbb{E}[\cos(Y)] &\simeq \cos(\mu)\left[1 - \frac{\mu^2(2\delta+5)}{2(\delta+1)^2}\right]\\ \mathbb{E}[\sin(\theta)] = \mathbb{E}[\sin(Y)] &\simeq \sin(\mu)\left[1 - \frac{\mu^2(2\delta+5)}{2(\delta+1)^2}\right].\end{aligned}$$

□

In circular distribution the interest, in general, is in the concentration parameter ρ given by $\rho = \sqrt{\mathbb{E}^2[\cos\theta] + \mathbb{E}^2[\sin\theta]}$. With the results given in Proposition 2 we have that the circular concentration parameter is given by

$$\rho = \sqrt{\mathbb{E}^2(\cos\theta) + \mathbb{E}^2(\sin\theta)} \simeq 1 - \frac{\mu^2(2\delta+5)}{2(\delta+1)^2}.$$

Remark 3. *As known* $0 < \rho < 1$, *so*

$$\begin{aligned} 0 < 1 - \frac{\mu^2(2\delta+5)}{2(\delta+1)^2} < 1 \qquad (2.4)\\ 0 < \mu < \frac{\sqrt{2}(\delta+1)}{\sqrt{2\delta+5}}.\end{aligned}$$

Definition 2. *Let* $\theta \sim \mathcal{WBS}(\mu, \delta)$, *then the cumulative distribution function of* θ *is given by*

$$\begin{aligned} F_\theta(\theta) &= \sum_{k=0}^{\infty} F(\theta + 2k\pi) - F(2k\pi) \qquad (2.5)\\ &= \sum_{k=0}^{\infty}\left\{\Phi\left(\sqrt{\frac{\delta}{2}}\left[\sqrt{\frac{(\delta+1)(\theta+2k\pi)}{\delta\mu}} - \sqrt{\frac{\delta\mu}{(\delta+1)(\theta+2k\pi)}}\right]\right)\right.\\ &\left. -\Phi\left(\sqrt{\frac{\delta}{2}}\left[\sqrt{\frac{(\delta+1)2k\pi}{\delta\mu}} - \sqrt{\frac{\delta\mu}{(\delta+1)2k\pi}}\right]\right)\right\},\end{aligned}$$

where F *is the cumulative distribution function of* $Y \sim \mathcal{BS}(\mu, \delta)$ *and* Φ *is the cumulative distribution function of the standard normal distribution, considering the relationship given in 2.1.*

3 Trigonometric Moments

Remark 4. *Let $Y \sim \mathcal{BS}(\mu,\delta)$, then its characteristic function is $\varphi : \mathbb{R} \to \mathbb{C}$, given by*

$$\varphi(t) = \mathbb{E}(\mathrm{e}^{itY}) = \frac{1}{2}\left\{\left[1 + \frac{\sqrt{\delta+1}}{\sqrt{1+\delta-4ti\mu}}\right]\exp\left(\frac{\delta\left[\sqrt{\delta+1}-\sqrt{1+\delta-4ti\mu}\right]}{2\sqrt{\delta+1}}\right)\right\},$$

where $t \in \mathbb{R}$ and $i = \sqrt{-1}$.

Definition 3. *Let $\theta \sim \mathcal{WBS}(\mu,\delta)$, then its characteristic function is $\varphi(p)$, $p \in \mathbb{Z}$ where φ is the characteristic function of $Y \sim \mathcal{BS}(\mu,\delta)$, that is, is the sequence $\{\varphi(0), \varphi(\pm 1), \varphi(\pm 2), \dots\}$ or $\{\varphi_0, \varphi_{\pm 1}, \varphi_{\pm 2}, \dots\}$. The value of the characteristic function at the integer p is called p^{th} trigonometric moment of θ.*

For $p = 0$ we have $\varphi_0 = \mathbb{E}(\mathrm{e}^0) = 1$, for $p = 1$ we have the first trigonometric moment of θ given by

$$\varphi_1 = \frac{1}{2}\left\{\left[1 + \frac{\sqrt{\delta+1}}{\sqrt{1+\delta-4i\mu}}\right]\exp\left(\frac{\delta\left[\sqrt{\delta+1}-\sqrt{1+\delta-4i\mu}\right]}{2\sqrt{\delta+1}}\right)\right\},$$

and for $p = 2$ we have the second trigonometric moment of θ given by

$$\varphi_2 = \frac{1}{2}\left\{\left[1 + \frac{\sqrt{\delta+1}}{\sqrt{1+\delta-8i\mu}}\right]\exp\left(\frac{\delta\left[\sqrt{\delta+1}-\sqrt{1+\delta-8i\mu}\right]}{2\sqrt{\delta+1}}\right)\right\}.$$

4 Maximum Likelihood Estimation

Let $t_1, \dots, t_n$ a random sample of $T \sim \mathcal{WBS}(\mu,\delta)$. Then the log-likelihood function is given by

$$\ell(\mu,\delta;t) = \frac{n}{2}(\delta + \log(\delta+1) - \log(16\pi\mu)) + \sum_{j=1}^{n}\log\left\{\sum_{k=0}^{\infty}\frac{\left(\frac{\delta\mu}{\delta+1} + t_j + 2k\pi\right)\exp\left\{-\frac{\delta}{4}\left(\frac{\delta\mu}{(\delta+1)(t_j+2k\pi)} + \frac{(\delta+1)(t_j+2k\pi)}{\delta\mu}\right)\right\}}{(t_j+2k\pi)^{3/2}}\right\}. \tag{4.1}$$

The Score function is given by $\mathbf{U}(\boldsymbol{\beta}) = (\mathbf{U}(\mu), \mathbf{U}(\delta))^\top$ where $\boldsymbol{\beta} = (\mu,\delta)^\top$

whereas

$$\mathbf{U}(\mu)=\frac{\partial\ell(\mu,\delta;t)}{\partial\mu}=-\frac{n}{2\mu}+\sum_{j=1}^{n}\left\{\left[\sum_{k=0}^{\infty}\frac{\delta\exp\left(-\frac{\delta}{4}\left(\frac{\delta\mu}{(\delta+1)\left(t_j+2k\pi\right)}+\frac{(\delta+1)\left(t_j+2k\pi\right)}{\delta\mu}\right)\right)}{(\delta+1)\left(t_j+2k\pi\right)^{3/2}}-\right.\right.$$

$$\left.\frac{\delta\left(\frac{\delta\mu}{\delta+1}+t_j+2k\pi\right)\left(\frac{\delta}{(\delta+1)\left(t_j+2k\pi\right)}-\frac{(\delta+1)\left(t_j+2k\pi\right)}{\delta\mu^2}\right)\mathrm{e}^{-\frac{\delta}{4}\left(\frac{\delta\mu}{(\delta+1)\left(t_j+2k\pi\right)}+\frac{(\delta+1)\left(t_j+2k\pi\right)}{\delta\mu}\right)}}{4\left(t_j+2k\pi\right)^{3/2}}\right]$$

$$\left.\times\left[\sum_{k=0}^{\infty}\frac{\left(\frac{\delta\mu}{\delta+1}+t_j+2k\pi\right)\exp\left(-\frac{1}{4}\delta\left(\frac{\delta\mu}{(\delta+1)\left(t_j+2k\pi\right)}+\frac{(\delta+1)\left(t_j+2k\pi\right)}{\delta\mu}\right)\right)}{\left(t_j+2k\pi\right)^{3/2}}\right]^{-1}\right\}$$

and

$$\mathbf{U}(\delta)=\frac{\partial\ell(\mu,\delta;t)}{\partial\delta}=\frac{n}{2}\left(\frac{1}{\delta+1}+1\right)+\sum_{j=1}^{n}\left\{\sum_{k=0}^{\infty}\left\{\left(\frac{\delta\mu}{\delta+1}+t_j+2k\pi\right)\right.\right.$$
$$\left(\frac{1}{4}\left(-\frac{\delta\mu}{(\delta+1)\left(t_j+2k\pi\right)}-\frac{(\delta+1)\left(t_j+2k\pi\right)}{\delta\mu}\right)-\right.$$
$$\left.\frac{\delta}{4}\left(-\frac{(\delta+1)\left(t_j+2k\pi\right)}{\delta^2\mu}+\frac{\mu}{(\delta+1)\left(t_j+2k\pi\right)}-\frac{\delta\mu}{(\delta+1)^2\left(t_j+2k\pi\right)}+\frac{t_j+2k\pi}{\delta\mu}\right)\right)$$
$$\exp\left(-\frac{\delta}{4}\left(\frac{\delta\mu}{(\delta+1)\left(t_j+2k\pi\right)}+\frac{(\delta+1)\left(t_j+2k\pi\right)}{\delta\mu}\right)\right)\left(t_j+2k\pi\right)^{-3/2}+$$
$$\left.\left.\frac{\left(\frac{\mu}{\delta+1}-\frac{\delta\mu}{(\delta+1)^2}\right)\exp\left(-\frac{1}{4}\delta\left(\frac{\delta\mu}{(\delta+1)(t_j+2k\pi)}+\frac{(\delta+1)(t_j+2k\pi)}{\delta\mu}\right)\right)}{\left(t_j+2k\pi\right)^{3/2}}\right\}\right\}\times$$
$$\left(\sum_{k=0}^{\infty}\frac{\left(\frac{\delta\mu}{\delta+1}+t_j+2k\pi\right)\exp\left(-\frac{\delta}{4}\left(\frac{\delta\mu}{(\delta+1)(t_j+2k\pi)}+\frac{(\delta+1)(t_j+2k\pi)}{\delta\mu}\right)\right)}{\left(t_j+2k\pi\right)^{3/2}}\right)^{-1}$$

The MLE has no closed-form expression for μ and δ. Therefore it is necessary numerical methods such the quasi-Newton BFGS method ([11]) to obtain MLE.

Let $\boldsymbol{\beta}=(\mu,\delta)^\top$ be the vector of parameters and let $\hat{\boldsymbol{\beta}}=(\hat{\mu},\hat{\delta})^\top$ denote its MLE. From the asymptotic normality of the MLE, it follows that

$$\hat{\boldsymbol{\beta}}\overset{A}{\sim}\mathrm{N}_2(\boldsymbol{\beta},\mathbf{K}(\boldsymbol{\beta})^{-1}),\tag{4.2}$$

when n is large, $\overset{A}{\sim}$ denoting approximately distributed. Here, $\mathbf{K}(\boldsymbol{\beta})$ is the Fisher's information matrix, $\mathbf{K}(\boldsymbol{\beta})^{-1}$ being its inverse, where

$$\mathbf{K}(\boldsymbol{\beta}) = \begin{bmatrix} \mathbf{K}(\boldsymbol{\beta})_{\mu\mu} & \mathbf{K}(\boldsymbol{\beta})_{\mu\delta} \\ \mathbf{K}(\boldsymbol{\beta})_{\delta\mu} & \mathbf{K}(\boldsymbol{\beta})_{\delta\delta} \end{bmatrix}, \quad \mathbf{K}(\boldsymbol{\beta})^{-1} = \begin{bmatrix} \mathbf{K}(\boldsymbol{\beta})^{\mu\mu} & \mathbf{K}(\boldsymbol{\beta})^{\mu\delta} \\ \mathbf{K}(\boldsymbol{\beta})^{\delta\mu} & \mathbf{K}(\boldsymbol{\beta})^{\delta\delta} \end{bmatrix},$$

$\mathbf{K}(\boldsymbol{\beta})_{\mu\mu} = \frac{\partial^2 \ell(\mu,\delta;y)}{\partial\mu\partial\mu}$, $\mathbf{K}(\boldsymbol{\beta})_{\mu\delta} = \frac{\partial^2 \ell(\mu,\delta;y)}{\partial\mu\partial\delta}$, $\mathbf{K}(\boldsymbol{\beta})_{\delta\mu} = \frac{\partial^2 \ell(\mu,\delta;y)}{\partial\delta\partial\mu}$, $\mathbf{K}(\boldsymbol{\beta})_{\delta\delta} = \frac{\partial^2 \ell(\mu,\delta;y)}{\partial\delta\partial\delta}$. In this work, $\mathbf{K}(\hat{\boldsymbol{\beta}})$ and $\mathbf{K}(\hat{\boldsymbol{\beta}})^{-1}$ were obtained numerically.

Therefore, from 4.2, one can construct asymptotic confidence intervals (ACIs) for $\boldsymbol{\beta}$, which is given by $\hat{\boldsymbol{\beta}} \pm z_{1-\alpha/2}\mathbf{K}(\hat{\boldsymbol{\beta}})^{-1/2}$, where $z_{1-\alpha/2}$ is the $(1-\alpha/2)^{\text{th}}$ quantile of the standard normal distribution and

$$\mathbf{K}(\hat{\boldsymbol{\beta}})^{-1/2} = \left(\sqrt{\mathbf{K}(\boldsymbol{\beta})^{\mu\mu}} \quad \sqrt{\mathbf{K}(\boldsymbol{\beta})^{\delta\delta}}\right)^{\top}.$$

For the circular concentration parameter ρ we have the ACI with confidence level $1-\alpha/2$

$$\hat{\rho} \pm z_{1-\alpha/2}\mathrm{Var}[d(\hat{\boldsymbol{\beta}})]^{1/2},$$

where $\hat{\rho} = d(\hat{\boldsymbol{\beta}}) = 1 - \dfrac{\hat{\mu}^2(2\hat{\delta}+5)}{2(\hat{\delta}+1)^2}$ and $\mathrm{Var}(\boldsymbol{\beta}) = \mathbf{K}(\boldsymbol{\beta})^{-1}$, so that $\mathrm{Var}[d(\hat{\boldsymbol{\beta}})] = D^{\top}(\hat{\boldsymbol{\beta}})\mathrm{Var}(\hat{\boldsymbol{\beta}})D(\hat{\boldsymbol{\beta}})$, where $D(\hat{\boldsymbol{\beta}}) = \left(\frac{\partial d(\hat{\boldsymbol{\beta}})}{\partial\hat{\mu}} \quad \frac{\partial d(\hat{\boldsymbol{\beta}})}{\partial\hat{\delta}}\right)^{\top}$.

5 Simulation Study

We conducted a simulation study to assess the performance of the maximum likelihood estimator (MLE). The number of Monte Carlo replication is 5000. In each of the 5000 replication we obtained the maximum likelihood estimates for values of μ and ρ given in the tables bellow. We calculated the mean square error (MSE), the relative bias (the relative bias of an estimator $\hat{\mu}$ of a parameter μ is defined as $\mathbb{E}(\hat{\mu}-\mu)/\mu$), the asymmetry (SKEW) and the kurtosis (KURT) of the MLE. We also obtain an asymptotic confidence interval (ACI) for the maximum likelihood estimator (MLE) in each iteration and calculate the ACI's coverage level. Note that δ were fixed as 42 to satisfy 2.4 in all cases. The sample size used were 15, 30, 45, 60 and 75. All simulations were carried out using the Ox matrix programming language. Ox is freely distributed for academic purposes and available at http://www.doornik.com. Birnbaum-Saunders random number generation was performed using the transformation 2.1.

The tables 1, 2, 3 and 4 shows that MSE of the MLE decreases as sample size increases and the coverage level of the asymptotic confidence intervals with confidence level of 95% for μ and ρ increases with the sample size in all cases. With small values of ρ and n the bias of $\hat{\rho}$ is relatively large but it decreases as sample size increases. The signal interchanging or the bias is an evidence that $\hat{\mu}$ is unbiased for μ and it is near to zero in all cases. The asymmetry and the kurtosis is near to those of the normal distribution values, as expected. The MSE of $\hat{\mu}$ and $\hat{\rho}$ increases as ρ decreases indicating that is more difficult to estimate the parameters when the data is less concentrated around a central angle.

Table 1: Monte Carlo simulation results for MLE with $\mu = \pi/4$ $(0,78539)$ and $\rho = 0,98515$

MAXIMUM LIKELIHOOD ESTIMATION						
n	$\hat{\mu}$	**BIAS %**	**MSE**	**SKEW**	**KURT**	**% ACI**
15	$0,78518$	$-2,737 \times 10^{-02}$	$1,945 \times 10^{-03}$	$0,1373$	$3,2122$	$91,68$
30	$0,78575$	$4,513 \times 10^{-02}$	$9,863 \times 10^{-04}$	$0,1868$	$3,2639$	$93,34$
45	$0,78524$	$-2,063 \times 10^{-02}$	$7,157 \times 10^{-04}$	$0,1229$	$3,0635$	$92,60$
60	$0,78573$	$4,266 \times 10^{-02}$	$4,990 \times 10^{-04}$	$0,1412$	$3,0117$	$93,92$
75	$0,78541$	$1,999 \times 10^{-03}$	$3,994 \times 10^{-04}$	$0,0680$	$2,9169$	$93,96$
n	$\hat{\rho}$	**BIAS %**	**MSE**	**SKEW**	**KURT**	**% ACI**
15	$0,98607$	$9,253 \times 10^{-02}$	$3,471 \times 10^{-05}$	$-1,0259$	$4,5583$	$84,34$
30	$0,98554$	$3,872 \times 10^{-02}$	$1,801 \times 10^{-05}$	$-0,6806$	$3,6922$	$89,06$
45	$0,98545$	$2,987 \times 10^{-02}$	$1,200 \times 10^{-05}$	$-0,5126$	$3,3268$	$90,44$
60	$0,98539$	$2,352 \times 10^{-02}$	$8,971 \times 10^{-06}$	$-0,5256$	$3,3762$	$91,64$
75	$0,98532$	$1,728 \times 10^{-02}$	$7,167 \times 10^{-06}$	$-0,4616$	$3,3474$	$92,82$

Table 2: Monte Carlo simulation results for MLE with $\mu = 3\pi/4$ $(2,35619)$ and $\rho = 0,86639$

MAXIMUM LIKELIHOOD ESTIMATION						
n	$\hat{\mu}$	**BIAS %**	**MSE**	**SKEW**	**KURT**	**% ACI**
15	$2,35555$	$-2,721 \times 10^{-02}$	$1,751 \times 10^{-02}$	$0,1378$	$3,2134$	$91,98$
30	$2,35725$	$4,476 \times 10^{-02}$	$8,878 \times 10^{-03}$	$0,1870$	$3,2646$	$93,74$
45	$2,35571$	$-2,073 \times 10^{-02}$	$6,442 \times 10^{-03}$	$0,1229$	$3,0635$	$92,94$
60	$2,35719$	$4,220 \times 10^{-02}$	$4,491 \times 10^{-03}$	$0,1416$	$3,0120$	$94,16$
75	$2,35623$	$1,499 \times 10^{-03}$	$3,594 \times 10^{-03}$	$0,0680$	$2,9179$	$94,18$
n	$\hat{\rho}$	**BIAS %**	**MSE**	**SKEW**	**KURT**	**% ACI**
15	$0,87460$	$9,481 \times 10^{-01}$	$2,816 \times 10^{-03}$	$-1,0317$	$4,5863$	$84,34$
30	$0,86985$	$3,994 \times 10^{-01}$	$1,460 \times 10^{-03}$	$-0,6833$	$3,6977$	$89,08$
45	$0,86904$	$3,066 \times 10^{-01}$	$9,726 \times 10^{-04}$	$-0,5126$	$3,3291$	$90,40$
60	$0,86850$	$2,441 \times 10^{-01}$	$7,276 \times 10^{-04}$	$-0,5277$	$3,3740$	$91,62$
75	$0,86795$	$1,801 \times 10^{-01}$	$5,806 \times 10^{-04}$	$-0,4621$	$3,3544$	$92,78$

Table 3: Monte Carlo simulation results for MLE with $\mu = 5\pi/4$ $(3,92699)$ and $\rho = 0,62886$

MAXIMUM LIKELIHOOD ESTIMATION						
n	$\hat{\mu}$	**BIAS %**	**MSE**	**SKEW**	**KURT**	**% ACI**
15	$3,92836$	$3,493 \times 10^{-02}$	$5,087 \times 10^{-02}$	$0,1992$	$3,3839$	$91,98$
30	$3,92995$	$7,543 \times 10^{-02}$	$2,564 \times 10^{-02}$	$0,1941$	$3,2867$	$93,66$
45	$3,92718$	$4,929 \times 10^{-03}$	$1,841 \times 10^{-02}$	$0,1658$	$3,1669$	$92,94$
60	$3,92939$	$6,115 \times 10^{-02}$	$1,268 \times 10^{-02}$	$0,1522$	$3,0304$	$94,26$
75	$3,92772$	$1,854 \times 10^{-02}$	$1,015 \times 10^{-02}$	$0,0884$	$2,9730$	$94,32$
n	$\hat{\rho}$	**BIAS %**	**MSE**	**SKEW**	**KURT**	**% IC**
15	$0,64878$	$3,1689$	$2,404 \times 10^{-02}$	$-1,3858$	$7,0027$	$84,36$
30	$0,63732$	$1,3467$	$1,183 \times 10^{-02}$	$-0,7923$	$4,1541$	$88,98$
45	$0,63537$	$1,0355$	$7,781 \times 10^{-03}$	$-0,5800$	$3,5305$	$90,48$
60	$0,63413$	$0,8383$	$5,745 \times 10^{-03}$	$-0,5632$	$3,5140$	$91,62$
75	$0,63268$	$0,6088$	$4,619 \times 10^{-03}$	$-0,5214$	$3,5385$	$92,76$

Table 4: Monte Carlo simulation results for MLE with $\mu = 7\pi/4$ $(5,49779)$ and $\rho = 0,27256$

MAXIMUM LIKELIHOOD ESTIMATION						
n	$\hat{\mu}$	**BIAS %**	**MSE**	**SKEW**	**KURT**	**% ACI**
15	$5,49311$	$-8,511 \times 10^{-02}$	$1,142 \times 10^{-01}$	$0,1072$	$2,9171$	$91,72$
30	$5,50371$	$1,078 \times 10^{-01}$	$6,096 \times 10^{-02}$	$0,3126$	$3,1850$	$93,64$
45	$5,50067$	$5,242 \times 10^{-02}$	$4,324 \times 10^{-02}$	$0,2974$	$3,1609$	$93,36$
60	$5,50295$	$9,383 \times 10^{-02}$	$3,071 \times 10^{-02}$	$0,2315$	$3,1074$	$94,10$
75	$5,49876$	$1,774 \times 10^{-02}$	$2,457 \times 10^{-02}$	$0,1724$	$3,1101$	$94,44$
n	$\hat{\rho}$	**BIAS %**	**MSE**	**SKEW**	**KURT**	**% ACI**
15	$0,33757$	$23,8523$	$6,535 \times 10^{-02}$	$-0,4157$	$2,5565$	$84,00$
30	$0,29565$	$8,4720$	$4,103 \times 10^{-02}$	$-0,3974$	$2,6628$	$88,36$
45	$0,28598$	$4,9264$	$3,074 \times 10^{-02}$	$-0,4049$	$2,7529$	$89,62$
60	$0,28123$	$3,1823$	$2,289 \times 10^{-02}$	$-0,3981$	$2,8736$	$91,50$
75	$0,27830$	$2,1077$	$1,970 \times 10^{-02}$	$-0,4127$	$2,9256$	$92,06$

6 An Application

Rudolf Jander's experiments concerning the direction chosen by ants in response to a stimulus has long provided some interesting problems in modeling of circular distributions (see [7] and [18]). The Figure 1 shows the rose diagram of the data and we can see that the values are concentrated around an angle near π.

An asymmetric version of the Von Mises distribution was developed by [18], it was used to modeling these data and already compared the obtained results with the Von Mises's model by Kuiper's statistics. These statistics is an extension of the well known Kolmogorov-Smirnov statistics for the real random variables case and it's definition is discussed in details in [6].

To check the goodness of fit of the WBS Model we computed the Kuiper's statistics and compared these value with those obtained by [18]. Table 5 displays the estimated values of μ, the mean of the distribution, for the three models, WBS, Von Mises and Asymmetric Von Mises and also displays the Kuiper's statistics for the three models. We can see that the smaller value of the Kuiper's statistics occurs for model WBS. So the WBS model provides a better fit to the these data than the others ones.

Table 5: Kuiper's Statistics and MLE

	MLE		
Parameter	WBS	VonMises	Asymmetric VonMises
μ	$3,7157$	$3,1963$	$2,8808$
ρ	$0,1809$	-	-
Kuiper's Statistics	$2,8888$	$12,5958$	$11,2918$

7 Conclusion Remarks

We present a new circular distribution called Wrapped Birnbaum-Saunders which is an asymmetric alternative for modeling circular data. Expressions were derived for its probability density function, cumulative distribution function and for its trigonometric moments. The maximum likelihood method was used to estimate its parameters and a Monte Carlo simulation study was carried out to evaluate the performance of the estimators. Finally, a real dataset was used to show the applicability of the proposed model and its performance was compared with that of the symmetric and asymmetric Von Mises models using Kuiper's statistics for circular data. It was concluded that the Wrapped Birnbaum-Saunders distribution presented satisfactory performance in the simulations and adjusted itself more appropriately to the application data than the other two models. Therefore, we note that the Wrapped Birnbaum-Saunders distribution is a promising model for circular asymmetric data.

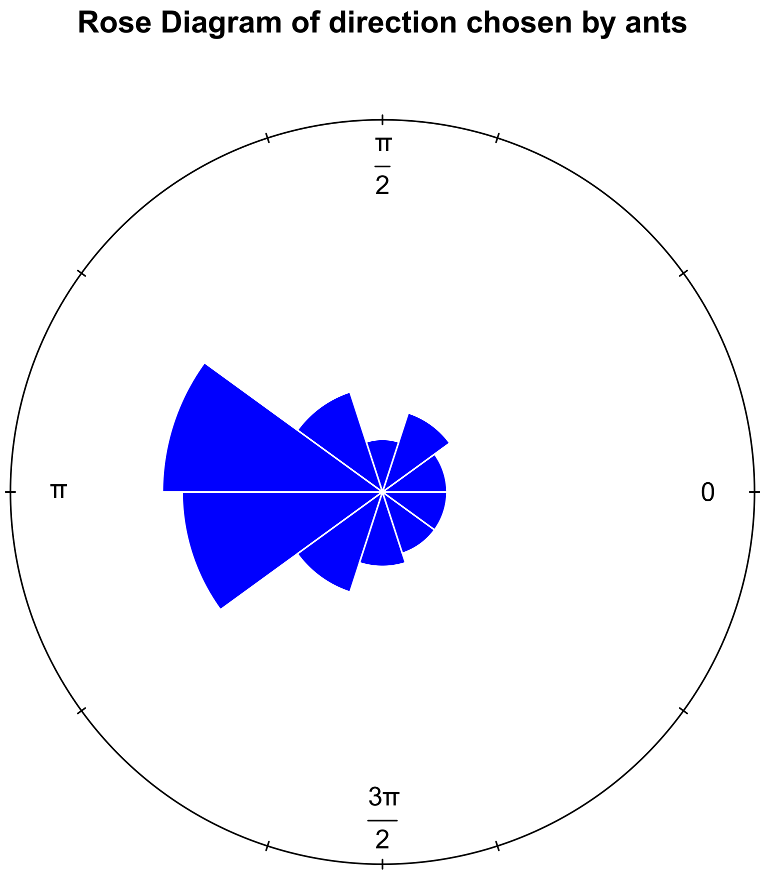

Figure 1: The rose diagram of the direction chosen by ants in the Rudolf Jander's experiment

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