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Hypersurfaces with radial mean curvature in space forms

Hipersuperfícies com curvatura média radial em formas espaciais

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Abstract: In this paper, we study two classes of hypersurfaces, namely, the DRMC-hypersurfaces and the HDRMC-hypersurfaces in space forms $\overline{M}^{n+1}(c)$, $c = -1, 0, 1$, these classes include the Weingarten hypersurfaces of the spherical type obtained in [10]. For $n = 2$, we present a way to obtain DRMC-surfaces and HDRMC-surfaces in $\overline{M}^3(c)$ using two holomorphic functions. Also, we classify the DRMC-hypersurfaces of rotation in $\overline{M}^{n+1}(c)$ and the HDRMC-hypersurfaces of rotation in $\mathbb{R}^{n+1}$.

Keywords: Weingarten hypersurfaces. radial mean curvature. Helmholtz equation.

Resumo: Neste artigo, estudamos duas classes de hipersuperficies, a saber, as DRMC-hipersuperfícies e as HDRMC-hipersuperfícies em formas espaciais $\overline{M}^{n+1}(c)$, $c = -1, 0, 1$, essas classes incluem as hipersuperfícies Weingarten de tipo esférico obtidas em [10]. Para $n = 2$, apresentamos uma forma de obter DRMC-superfícies e HDRMC-superfícies em $\overline{M}^3(c)$ usando duas funções holomorfas. Também classificamos as DRMC-hipersuperfícies de rotação em $\overline{M}^{n+1}(c)$ e as HDRMC-hipersuperfícies de rotação em $\mathbb{R}^{n+1}$.

Palavras-chave: Hipersuperfícies Weingarten. curvatura média radial. equação de Helmholtz.

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1 Introduction

The surfaces $M \subset \mathbb{R}^3$ satisfying a functional relation of the form $W(H, K) = 0$, where H and K are the mean and Gaussian curvatures of the surface M, respectively, are called *Weingarten surfaces*. Examples of Weingarten surfaces are the surfaces of revolution and the surfaces of constant mean or Gaussian curvature. In [7], the authors study an important class of surfaces satisfying a linear relation of the form

$$aH + bK + c = 0,$$

where $a, b, c \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. These surfaces are called *linear Weingarten surfaces*. The paper [6], is devoted to the integrability of linear Weingarten surfaces.

Corro, in [2] presented a way of parameterizing surfaces as envelopes of a congruence of spheres in which an envelope is contained in a plane and with radius function h associated with a hydrodynamic type system. As an application, it studies the surfaces in hyperbolic space $\mathbb{H}^3$ satisfying the relation

$$2ach^{\frac{2(c-1)}{c}}(H - 1) + (a + b - ach^{\frac{2(c-1)}{c}})K = 0,$$

where $a, b, c \in \mathbb{R}$, $a + b \neq 0$, $c \neq 0$, H is the mean curvature and K is the Gaussian curvature. This class of surfaces includes the Bryant surfaces and the flat surfaces of the hyperbolic space and are called *generalized Weingarten surfaces of Bryant type*.

In [3] the authors study the surfaces M in the hyperbolic space $\mathbb{H}^3$ satisfying the relation

$$2(H - 1)e^{2\mu} + K(1 - e^{2\mu}) = 0,$$

where μ is a harmonic function with respect to the quadratic form $\sigma = -KI + 2(H-1)II$, I and II are the first and the second quadratic form of M. These surfaces are called *Generalized Weingarten surfaces of harmonic type*.

In [5], the authors study a class of oriented hypersurfaces M in hyperbolic space $(n + 1)$-dimensional that satisfy a Weingarten relation in the form

$$\sum_{r=0}^{n}(c - n + 2r)\binom{n}{r}H_r = 0,$$

where c is a real constant and H_r is the rth mean curvature of the hypersurface M. They show that this class of hypersurfaces is characterized by a harmonic application derived from the two hyperbolic Gauss map. Looking these hypersurfaces as orthogonal to a congruence of geodesics, they also show the relation of such hypersurfaces with solutions of the equation $\Delta u + ku^{\frac{n+2}{n-2}} = 0$, where $k \in \{-1, 0, 1\}$.

In [9], the author present a way to parameterize hypersurfaces as congruence of spheres in which an envelope is contained in a hyperplane. Using this parametrization is presented a generalization of the surfaces of the spherical type (Laguerre minimal surfaces) studied in [8], namely the *Weingarten hypersurfaces of the spherical type*, i.e. the oriented hypersurfaces of the Euclidean space $M \subset \mathbb{R}^{n+1}$ satisfying a Weingarten relation of the form

$$\sum_{r=1}^{n}(-1)^{r+1} r f^{r-1}\binom{n}{r} H_r = 0,$$

where $f \in C^{\infty}(M;\mathbb{R})$ and H_r is the rth mean curvature of M. Later, Reyes and Riveros [10], generalize the results obtained by [9] in space forms.

In this paper, we study two classes of hypersurfaces, namely, the DRMC-hypersurfaces and the HDRMC-hypersurfaces in space forms $\overline{M}^{n+1}(c)$, $c = -1, 0, 1$, defined as:
An orientable hypersurface $M \subset \mathbb{R}^{n+1}, n \geq 2$, is called a *hypersurface with radial mean curvature which depends on the distance and radius functions* (in short, DRMC-hypersurface) if satisfy

$$\frac{H_R}{1-d} + (a-c)h = 0,\ a \in \mathbb{R}.$$

An orientable hypersurface $M \subset \mathbb{R}^{n+1}, n \geq 2$, is called a *hypersurface with radial mean curvature of harmonic type* (in short HDRMC-hypersurface) if satisfy

$$\Delta\left(\frac{H_R}{1-d} + (a-c)h\right) = 0,$$

where H_R is the radial mean curvature.
We observe that when $a = c = 0$ and $H_R = 0$ we obtain the Weingarten hypersurfaces of the spherical type estudied by Machado in [9], also, when $a = c$ and $H_R = 0$ we obtain the Weingarten hypersurfaces of the spherical type estudied by Reyes and Riveros in [10]. For $n = 2$ we present a way to obtain DRMC-surfaces and HDRMC-surfaces in $\overline{M}^3(c)$ using two holomorphic functions. Also, we classify the DRMC-hypersurfaces of rotation in $\overline{M}^{n+1}(c)$ and the HDRMC-hypersurfaces of

rotation in $\mathbb{R}^{n+1}$.

2 Preliminaries

Let $\overline{M}^{n+1}(c)$ be, the simply connected space form of sectional curvature $c = -1, 1, 0$. $\overline{M}^{n+1}(c)$ will denote the (n+1)-dimensional hyperbolic space $\mathbb{H}^{n+1}$, if $c = -1$, the Euclidean space $\mathbb{R}^{n+1}$ when $c = 0$ or the sphere $\mathbb{S}^{n+1}$, if $c = 1$.
Let $U \subset \mathbb{R}^n$ be an open set of $\mathbb{R}^n$ such that $u = (u_1, u_2, \ldots, u_n) \in U$. The partial derivatives of $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$, with respect to u_i, $1 \leq i \leq n$, will be denoted by $f_{,i}$.
We denote by $\mathbb{L}^{n+2}$ the space of (n+2)-tuples $u = (u_1, u_2, \ldots, u_{n+2}) \in \mathbb{R}^{n+2}$ with the Lorentzian metric $\langle u, v\rangle = \sum_{i=1}^{n+1} u_i v_i - u_{n+2}v_{n+2}$, where $v = (v_1, v_2, \ldots, v_{n+2})$ and we consider the hyperbolic space $\mathbb{H}^{n+1}$ as a hypersurface of $\mathbb{L}^{n+2}$, namely,

$$\mathbb{H}^{n+1} = \left\{u \in \mathbb{L}^{n+2}; \langle u, u\rangle = -1, u_{n+2} > 0\right\}.$$

Also, we consider the sphere $\mathbb{S}^{n+1}$ as a hypersurface of $\mathbb{R}^{n+2}$ with the Euclidean metric, namely,

$$\mathbb{S}^{n+1} = \left\{u \in \mathbb{R}^{n+2}; \langle u, u\rangle = 1\right\}.$$

Definition 1. Let M be a hypersurface of $\overline{M}^{n+1}(c)$. We say that M is *orientable*, if there exist a unit vector field N normal to T_pM, for all $p \in M$. N is known as *Gauss map* of M. In local coordinates,

$$N_{,i} = \sum_{j=1}^{n} W_{ij} X_{,j},\ 1 \leq i \leq n,$$

where X is a parametrization of M. The matrix $W = (W_{ij})$ is known as *Weingarten matrix* of M.

Definition 2. The *mean curvature* and the *Gauss-Kronecker curvature* of M are given by

$$H = \frac{1}{n}\sum_{i=1}^{n} k_i\ ,\ K = \prod_{i=1}^{n} k_i,$$

where $k_1, \ldots, k_n$ are the principal curvatures of M.

Definition 3. The *rth-mean curvature* H_r of M is defined by

$$H_r = \frac{S_r(W)}{\binom{n}{r}},$$

where, for intergers $0 \leq r \leq n$, $S_r(W)$, is defined by

$$\begin{array}{rcl} S_0(W) & = & 1, \\ S_r(W) & = & \displaystyle\sum_{1 \leq i_1 < \ldots < i_r \leq n} k_{i_1} \ldots k_{i_r}. \end{array}$$

Definition 4. Let M be a hypersurface of $\overline{M}^{n+1}(c)$, $n \geq 2$. M is a *Weingarten hypersurface of the spherical type* in $\overline{M}^{n+1}(c)$, if the rth mean curvatures of M in $\overline{M}^{n+1}(c)$ satisfy the equation

$$\sum_{r=1}^{n} (-1)^{r-1} r f^{r-1} H_r = 0,$$

for some function $f \in C^\infty(M, \mathbb{R})$.

From now on, we will consider e_c given by

$$e_c = \begin{cases} (0, 0, \ldots, 0, 1, 0) \in \mathbb{L}^{n+2}, \text{ if } c = -1, \\ (0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+1}, \text{ if } c = 0, \\ (0, 0, \ldots, 0, 0, 1) \in \mathbb{R}^{n+2}, \text{ if } c = 1. \end{cases}$$

Definition 5. Let M be an orientable hypersurface in $\overline{M}^{n+1}(c)$ and N the unit normal vector field of M in $\overline{M}^{n+1}(c)$, such that $N(p) \neq e_c$, $\forall\, p \in M$. We define the *distance* and *radius functions* $d, h : M \to \mathbb{R}$ given by

$$d(p) = \langle N(p), e_c \rangle, \quad h(p) = \frac{\langle p, e_c \rangle}{1 - d}, p \in M \tag{1}$$

and the *radial curvature* $\overline{k}_i$ of M as

$$\overline{k}_i = \frac{k_i}{h k_i - 1}, \; 1 \leq i \leq n, \tag{2}$$

with $hk_i - 1 \neq 0$, $\forall\; 1 \leq i \leq n$ and k_i are the principal curvatures of $M \subset \overline{M}^{n+1}(c)$.

Definition 6. We define the *radial mean curvature* H_R of the hypersurface M in $\overline{M}^{n+1}(c)$ as

$$H_R = \frac{1}{n}\sum_{i=1}^{n} \overline{k}_i. \tag{3}$$

We consider $M^n(c)$ a hypersurface of $\overline{M}^{n+1}(c)$, such that $M^n(c) = \mathbb{H}^n$, if $c = -1$, $M^n(c) = \mathbb{R}^n$ if $c = 0$ or $M^n(c) = \mathbb{S}^n$, if $c = 1$, with unit normal vector field $N(p) = e_c$, $\forall\, p \in M^n(c)$.
Let $Y : U \to M^n(c)$ be a local orthogonal parametrization of $M^n(c)$. If $L_{ij} = \langle Y_{,i}, Y_{,j}\rangle, 1 \leq i, j \leq n$, then $L_{ii} \neq 0$ and $L_{ij} = 0$ for $i \neq j$. The Christoffel symbols of L_{ij} are given by

$$\Gamma_{ij}^{m} = 0, \text{ for distinct } i,j,m, \ \Gamma_{ij}^{j} = \frac{L_{jj,i}}{2L_{jj}}, \text{ for all } i,j, \ \Gamma_{ii}^{j} = -\frac{L_{ii,j}}{2L_{jj}}, \text{ for } i \neq j. \tag{4}$$

We consider the sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$, $e_1 = (0,0,\ldots,0,1)$ and $-e_1 = (0,0,\ldots,0,-1)$ the north pole and south pole of $\mathbb{S}^{n+1}$, respectively. The stereographic projection $P_- : \mathbb{S}^{n+1} - \{-e_1\} \to \mathbb{R}^{n+1}$ and $P_+ : \mathbb{S}^{n+1} - \{e_1\} \to \mathbb{R}^{n+1}$ are diffeomorphism given by

$$P_-(q) = \frac{q - \langle q, e_1\rangle e_1}{1 + \langle q, e_1\rangle}, \ P_+(q) = \frac{q - \langle q, e_1\rangle e_1}{1 - \langle q, e_1\rangle}, \ q \in \mathbb{S}^{n+1}. \tag{5}$$

Therefore, the inverse mapping P_-^{-1} and P_+^{-1} are given by

$$P_-^{-1}(p) = \frac{(2p, 1 - \langle p, p\rangle)}{1 + \langle p, p\rangle}, \ P_+^{-1}(p) = \frac{(2p, \langle p, p\rangle - 1)}{1 + \langle p, p\rangle}, \ p \in \mathbb{R}^{n+1}. \tag{6}$$

We consider $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ and we define

$$\begin{aligned} P \ &: \ \mathbb{H}^{n+1} \to \mathbb{R}^{n+1} \\ &\quad u \to P(u), \end{aligned} \tag{7}$$

where $P(u)$ is the intersection of the hyperplane

$$\mathbb{R}^{n+1} = \left\{(u_1, u_2, \ldots, u_{n+1}, u_{n+2}) \subset \mathbb{R}^{n+2}; u_{n+2} = 0\right\}$$

with the line that passes through the points u and $(0,0,\ldots,0,-1) \in \mathbb{R}^{n+2}$. P is known as the *hyperbolic stereographic projection.*

The following result was obtained in [1].

Proposition 1. *Let $P : \mathbb{H}^{n+1} \to \mathbb{R}^{n+1}$ be given by (7). Then P is a diffeomorphism of $\mathbb{H}^{n+1}$ on $B^{n+1}(1) = \{u \in \mathbb{R}^{n+1}; |u| < 1\}$.*
Therefore, $P^{-1} : B^{n+1}(1) \to \mathbb{H}^{n+1}$ given by

$$P^{-1}(u) = \frac{1}{1 - \langle u, u\rangle}\left(2u, 1 + \langle u, u\rangle\right), \; u \in B^{n+1}(1), \tag{8}$$

is a parametrization of $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$.

The following results were obtained in [10].

Theorem 1. *Consider Σ an orientable hypersurface of $\overline{M}^{n+1}(c)$, N the unit normal vector field of Σ in $\overline{M}^{n+1}(c)$ such that $N(p) \neq e_c$, $\forall\, p \in \Sigma$, $h : \Sigma \to \mathbb{R}$ given by (1) and $X : U \to \Sigma$ a local parametrization of $p \in \Sigma$. Then, there exist a local parametrization $Y : U \to M^n(c)$, such that*

$$X(u) = Y(u) + h(u)\left[e_c - N(u)\right], \; u \in U. \tag{9}$$

If Y is a local orthogonal parametrization of $M^n(c)$, then

$$X = Y - \frac{2h}{S}\left(\sum_{i=1}^{n} \frac{h_{,i}}{L_{ii}} Y_{,i} - e_c + chY\right), \tag{10}$$

$$N = \frac{2}{S}\left(\sum_{i=1}^{n} \frac{h_{,i}}{L_{ii}} Y_{,i} - e_c + chY\right) + e_c, \tag{11}$$

where

$$S = \sum_{i=1}^{n} \frac{(h_{,i})^2}{L_{ii}} + ch^2 + 1 \neq 0. \tag{12}$$

The first, second and third fundamental forms of Σ *in* $\overline{M}^{n+1}(c)$*, are given by*

$$\begin{aligned} I = \langle X_{,i}, X_{,j} \rangle &= L_{ij} - \frac{2h}{S}(V_{ji}L_{ii} + V_{ij}L_{jj}) + \frac{4h^2}{S^2}\sum_{k=1}^{n} V_{ik}V_{jk}L_{kk}, \quad (13) \\ II = -\langle N_{,i}, X_{,j} \rangle &= \frac{4h}{S^2}\sum_{k=1}^{n} V_{ik}V_{jk}L_{kk} - \frac{2}{S}V_{ji}L_{ii}, \quad (14) \\ III = \langle N_{,i}, N_{,j} \rangle &= \frac{4}{S^2}\sum_{k=1}^{n} V_{ik}V_{jk}L_{kk}, \quad (15) \end{aligned}$$

respectively, where

$$V_{ij} = \frac{1}{L_{jj}}\left(h_{,ij} - \sum_{l=1}^{n} \Gamma_{ij}^{l} h_{,l} \right) + ch\delta_{ij}, \quad 1 \leq i, j \leq n, \quad (16)$$

and Γ_{ij}^{l} *are the Christoffel symbols of the metric* $L_{ij} = \langle Y_{,i}, Y_{,j} \rangle$, $1 \leq i, j \leq n$*.*
The Weingarten matrix $W = (W_{ij})$ *is given by*

$$W = 2(SI_n - 2hV)^{-1}V, \quad (17)$$

where I_n *is the identity matrix and* $V = (V_{ij})$*.*
The condition of regularity of X *is given by*

$$det(SI_n - 2hV) \neq 0. \quad (18)$$

Conversely, given a local orthogonal parametrization $Y : U \to M^n(c) \subset \overline{M}^{n+1}(c)$*, where* U *is a simply connected domain of* $\mathbb{R}^n$ *and a differentiable function* $h : U \to \mathbb{R}$*. Then (10) is a hypersurface of* $\overline{M}^{n+1}(c)$ *with Gauss map given by (11) and (12)-(18) are satisfied.*

Proposition 2. *Let* $X : U \subset \mathbb{R}^n \to \Sigma \subset \overline{M}^{n+1}(c)$ *be a parametrization of a hypersurface* Σ *given by (10). The following statements are equivalent*

(1) X *is parametrized by lines of curvature.*

(2) $V_{ij} = 0$*, for* $1 \leq i \neq j \leq n$*.*

(3) $N_{,i} = -k_{,i}X_{,i}$*, for all* $1 \leq i \leq n$*, where*

$$k_i = \frac{2V_{ii}}{2hV_{ii} - S}, \quad 1 \le i \le n, \tag{19}$$

are the principal curvatures of X.

Remark 1. From (19), the eigenvalues σ_i of the matrix V are given by

$$\sigma_i = \frac{Sk_i}{2(hk_i - 1)}, \quad 1 \le i \le n, \tag{20}$$

where k_i are the eigenvalues of the Weingarten matrix W.
From (20) we have that $\sigma_i = \frac{S}{2}\overline{k}_i$. Therefore,

$$\sum_{i=1}^{n} V_{ii} = \frac{nS}{2} H_R. \tag{21}$$

Let Y be a local orthogonal parametrization of $M^n(c) \subset \overline{M}^{n+1}(c)$ given by

$$Y = \begin{cases} P_-^{-1}, P_+^{-1} : \mathbb{R}^n \to \mathbb{S}^n, \text{ if } c = 1, \\ I : \mathbb{R}^n \to \mathbb{R}^n, \text{ if } c = 0, \\ P^{-1} : B^n(1) \to \mathbb{H}^n, \text{ if } c = -1, \end{cases} \tag{22}$$

where P_-^{-1}, P_+^{-1} are given by (6), I is the identity function of $\mathbb{R}^n$ and P^{-1} is given by (8). The metric L in the parametrization Y is given by $L_{ij} = \langle Y_{,i}, Y_{,j} \rangle = 0$, if $1 \le i \ne j \le n$ and $L_{ii} = \langle Y_{,i}, Y_{,i} \rangle = J_c$, where

$$J_c(u) = \begin{cases} \frac{4}{(1+\langle u,u\rangle)^2}, \ u \in \mathbb{R}^n, \text{if } c = 1, \\ 1, \ u \in \mathbb{R}^n, \text{if } c = 0, \\ \frac{4}{(1-\langle u,u\rangle)^2}, u \in B^n(1), \text{ if } c = -1. \end{cases} \tag{23}$$

From (4), the Christoffel symbols associated to L_{ij} are given by

$$\Gamma_{ii}^i = \frac{J_{c,i}}{2J_c}, \quad \Gamma_{ij}^i = \frac{J_{c,j}}{2J_c} = -\Gamma_{ii}^j, \quad 1 \le i \ne j \le n. \tag{24}$$

The following result can be found in [10].

Theorem 2. *Let Σ be an orientable hypersurface of $\overline{M}^{n+1}(c)$ given by Theorem 1 where Y is the local orthogonal parametrization of $M^n(c) \subset \overline{M}^{n+1}(c)$ given by (22). Σ is a rotation spherical hypersurface of $\overline{M}^{n+1}(c)$ if and only if h is a radial function.*

In [4], is introduced the generalized Helmholtz equation and present explicit solutions to this generalized Helmholtz equation, these solutions depend on three holomorphic functions.
The *two-dimensional Helmholtz equation* for $h : U \subset \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\Delta h(u) + k\Omega^2(u)h(u) = 0, \tag{25}$$

where $\Omega(u)$ indicates the wave number and k is a non-zero real constant.

Definition 7. The *two-dimensional generalized Helmholtz equation* for $h : U \subset \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$\Delta \left[\frac{1}{\Omega^2(u)} \left(\Delta h(u) + k\Omega^2(u)h(u) \right) \right] = 0, \tag{26}$$

where $\Omega(u)$ is a non-zero C^2 function and k is a non-zero real constant.

The following Lemma is an equivalent version to Lemma 1 shown in [11].

Lemma 1. *If $f_1, f_2, g : \mathbb{C} \to \mathbb{C}$ are holomorphic functions of $u = u_1 + iu_2$, such that $\langle 1, f_1 \rangle + \langle g, f_2 \rangle = 0$. Then $f_1 = -\overline{z}_1 g + ic_1$, $f_2 = ic_2 g + z_1$, where c_i are real constants and $z_1 \in \mathbb{C}$.*

3 Hypersurfaces with radial mean curvature

In this section, we study two classes of hypersurfaces, namely, the DRMC-hypersurfaces and the HDRMC-hypersurfaces.

Definition 8. We say that M is a *hypersurface with radial mean curvature which depends on the distance and radius functions* (in short DRMC-hypersurface) if the relation

$$\frac{H_R}{1-d} + (a-c)h = 0, \; a \in \mathbb{R}, \tag{27}$$

is satisfied.
Also, we say that M is a *hypersurface with radial mean curvature of harmonic type* (in short HDRMC-hypersurface) if the relation

$$\Delta \left(\frac{H_R}{1-d} + (a-c)h \right) = 0, \tag{28}$$

is satisfied.

We observe that when $H_R = 0$, M is a Weingarten hypersurface of the spherical type in $\overline{M}^{n+1}(c)$ (see [10] for more details).

Proposition 3. *Let Σ be an orientable hypersurface of $\overline{M}^{n+1}(c)$ given by Theorem 1, where Y is a local orthogonal parametrization of $M^n(c) \subset \overline{M}^{n+1}(c)$. Then Σ defines a hypersurface in $\overline{M}^{n+1}(c)$ satisfying*

$$\Delta_L h + nch = \frac{n}{1-d} H_R, \tag{29}$$

where L is the metric of $M^n(c)$ given by $L_{ij} = \langle Y_{,i}, Y_{,j} \rangle$, $1 \leq i, j \leq n$ and Δ_L is the Laplacian operator with respect to the metric L.

Proof. Let Σ be an orientable hypersurface of $\overline{M}^{n+1}(c)$ given by Theorem 1. From (16) we obtain that the trace of the matrix V in terms of the Laplacian operator is given by

$$\sum_{i=1}^{n} V_{ii} = \Delta_L h + nch. \tag{30}$$

From (11), we get $d = \langle N(p), e_c \rangle = 1 - \frac{2}{S}$, hence, $S = \dfrac{2}{1-d}$. Using (30) in (21) we obtain (29). The proof is complete. □

Corollary 1. *Let Σ be an orientable hypersurface of $\overline{M}^{n+1}(c)$ given by Theorem 1 and $a \in \mathbb{R}$.*

(1) *Σ is DRMC-hypersurface if and only if $\triangle_L h + nah = 0$.*

(2) *Σ is HDRMC-hypersurface if and only if $\Delta(\triangle_L h + nah) = 0$.*

(3) *A DRMC-hypersurface Σ in $\overline{M}^{n+1}(c)$ with $h \neq 0$ is a Weingarten hypersurface of the spherical type if and only if $a = c$.*

Proof. By Proposition 3, we get

$$\triangle_L h + nch = \frac{n}{1-d} H_R \Longleftrightarrow \triangle_L h + nah = \frac{n}{1-d} H_R + n(a-c)h.$$

Therefore,

$$\triangle_L h + nah = 0 \Longleftrightarrow \frac{1}{1-d} H_R + (a-c)h = 0,$$

$$\Delta\left(\triangle_L h + nah\right) = 0 \Longleftrightarrow \Delta\left(\frac{1}{1-d}H_R + (a-c)h\right) = 0.$$

From these expressions we get (1) and (2).
(3) If Σ is a Weingarten hypersurface of the spherical type then $H_R = 0$ and consequently $(a-c)h = 0$, therefore $a = c$.
Conversely, if $a = c$ then $\frac{1}{1-d}H_R = 0$. The proof is complete. □

Theorem 3. *Let Σ be an orientable hypersurface of $\overline{M}^{n+1}(c)$, $n \geq 2$ given by Theorem 1 where Y is the local orthogonal parametrization of $M^n(c) \subset \overline{M}^{n+1}(c)$ given by (22). Then Σ is a DRMC-hypersurface or a HDRMC-hypersurface if and only if h is a solution of the equation given by*

$$\frac{\Delta h}{J_c} + \frac{(n-2)}{2(J_c)^2}\langle \nabla J_c, \nabla h\rangle + anh = 0, \tag{31}$$

$$\Delta\left[\frac{\Delta h}{J_c} + \frac{(n-2)}{2(J_c)^2}\langle \nabla J_c, \nabla h\rangle + anh\right] = 0, \tag{32}$$

respectively, where J_c is given by (23).

Proof. By Corollary 1, we must calculate $\Delta_L h$ (the Laplacian operator of the function h with respect to the metric L) in the parameterization Y given by (22). From Remark 1 we have that $L_{ij} = 0$, if $1 \leq 1 \neq j \leq n$ and $L_{ii} = J_c, 1 \leq i \leq n$. Thus, from definition of Laplacian operator we obtain that

$$\Delta_L h = \frac{\Delta h}{J_c} + \frac{(n-2)}{2(J_c)^2}\langle \nabla J_c, \nabla h\rangle .$$

Hence, it follows (31) and (32). □

Remark 2. For $n = 2$, from Theorem 3 we obtain that the DRMC-surfaces and the HDRMC-surfaces satisfy

$$\frac{1}{J_c}\left(\Delta h + 2aJ_c h\right) = 0, \tag{33}$$

$$\Delta\left[\frac{1}{J_c}\left(\Delta h + 2aJ_c h\right)\right] = 0, \tag{34}$$

respectively.

In the following result we present a way to obtain DRMC-surfaces and HDRMC-surfaces in $\overline{M}^3(c)$ using two holomorphic functions.

Corollary 2. *On the conditions of the Theorem3.*
i) For $n = 2, a = 1, c = \pm 1$,

(1) *the solutions of (34) are given by* $h = \dfrac{\langle 1, A\rangle + \langle u, B\rangle}{1 + c\,|u|^2}$, *where* A, B *are holomorphic functions,*

(2) *the solutions of (33) are given by* $h = \dfrac{\langle 1, A\rangle + \langle u, B\rangle}{1 + c\,|u|^2}$, *where* A *is a holomorphic function and* B *is a holomorphic function such that* $B = \displaystyle\int (cA'u - cA + ic_1)\, du$, c_1 *is a real constant.*

ii) For $n = 2, a \in \mathbb{R}, a \neq 0, c = 0$,

(3) *some solutions of (33) are given by*

$$h(u) = \frac{\Omega C_1 C_2}{a} e^{-\left(\frac{c_1 - 2ac_2}{2|z_1|^2}\right)(b_1u_1 + a_1u_2)} \sin\left(\frac{\Omega}{2|z_1|^2}(a_1u_1 - b_1u_2)\right), \tag{35}$$

(4) *some solutions of (34) are given by*

$$\begin{aligned} h(u) \;=\; & -\frac{1}{2a^2|z_1|^2} e^{-2\alpha(b_1u_1 + a_1u_2)} \Big(C_2C_3K_1 e^{\alpha(b_1u_1 + a_1u_2)}(a_1\cos(\alpha(a_1u_1 - b_1u_2)) \\ & + b_1 \sin(\alpha(a_1u_1 - b_1u_2))) + C_1C_3K_2 e^{\left(\alpha + \frac{\Omega}{2|z_1|^2}\right)}(a_1\cos(\beta(a_1u_1 - b_1u_2)) \\ & + b_1\sin(\beta(a_1u_1 - b_1u_2))) + 4|z_1|^2 C_1C_2 e^{\frac{\Omega}{2|z_1|^2}(b_1u_1 + a_1u_2)} \times \\ & \sin\left(\frac{\Omega}{2|z_1|^2}(a_1u_1 - b_1u_2)\right)\Big), \end{aligned} \tag{36}$$

where
$c_1, c_2, C_1, C_2, C_3 \in \mathbb{R}, z_1 = a_1 + ib_1 \in \mathbb{C}, \Omega = \sqrt{c_1^2 + 4a(2|z_1|^2 - c_1c_2 + ac_2^2)}$,
$\alpha = \frac{c_1 - 2ac_2 + \Omega}{4|z_1|^2}$, $\beta = \frac{c_1 - 2ac_2 - \Omega}{4|z_1|^2}$, $K_1 = a(4|z_1|^2 - 2c_1c_2) + c_1(c_1 - \Omega)$,
$K_2 = a(4|z_1|^2 - 2c_1c_2) + c_1(c_1 + \Omega)$.

Proof. i) We will show that the given a holomorphic function g, non-zero real constants r, s with $k = rs$ and $\Omega(u) = \dfrac{2\sqrt{2}|g'|}{r+s|g|^2}$, the functions

$$h(u) = \frac{\langle 1, A\rangle + \langle g, B\rangle}{r + s\,|g|^2}, \tag{37}$$

are solutions of the two-dimensional generalized Helmholtz equation (26), where A and B are holomorphic functions.
Moreover, (37) are solutions of the two-dimensional Helmholtz equation (25) if the holomorphic functions A and B satisfy

$$B(u) = \frac{1}{r}\int (sgA' - sg'A + ic_1 g')du. \tag{38}$$

Consider

$$h = \frac{f}{T}, \quad \text{where } T = r + s|g|^2. \tag{39}$$

Calculating the Laplacian of h we have

$$\Delta h = \frac{\Delta f}{T} + 2\left\langle \nabla f, \nabla\left(\frac{1}{T}\right)\right\rangle + f\Delta\left(\frac{1}{T}\right).$$

Using the expression of T given in (39), we get

$$\begin{aligned} \Delta h &= \frac{\Delta f}{T} - 4s\left\langle \nabla f, \frac{g\overline{g'}}{T^2}\right\rangle + f\left(-\frac{4s|g'|^2}{T^2} + \frac{8s^2|gg'|^2}{T^3}\right) \\ &= \frac{\Delta f}{T} - 4\left\langle \nabla f, \frac{g\overline{g'}}{T^2}\right\rangle + 4fs|g'|^2\left(\frac{1}{T^2} - \frac{2r}{T^3}\right). \end{aligned}$$

This equation can be written as

$$\frac{T^2}{|g'|^2}\left(\Delta h + \frac{8rs|g'|^2}{T^2}h\right) = T\frac{\Delta f}{|g'|^2} - 4s\left\langle \nabla f, \frac{g}{g'}\right\rangle + 4sf. \tag{40}$$

Thus, for $\Omega = \dfrac{2\sqrt{2}|g'|}{r+s|g|^2}$, the function $h = \dfrac{f}{T}$ is a solution of the generalized Helmholtz equation (26), if and only if

$$\Delta\left\{T\frac{\Delta f}{|g'|^2} - 4s\left\langle \nabla f, \frac{g}{g'}\right\rangle + 4sf\right\} = T\Delta\left(\frac{\Delta f}{|g'|^2}\right) = 0.$$

On the other hand, the solutions of the equation $\Delta\left(\dfrac{\Delta f}{|g'|^2}\right) = 0$ are given by $f = \langle 1, A\rangle + \langle g, B\rangle$, where A, B are holomorphic functions. Thus, we get (37).
Also, it is easy to show that (40) is equivalent to

$$\left\langle 1, r\frac{B'}{g'} - s\frac{gA'}{g'} + sA\right\rangle = 0.$$

From this expression we obtain

$$r\frac{B'}{g'} - s\frac{gA'}{g'} + sA = ic_1.$$

Hence, we get (38). Therefore, (1) and (2) follows from (33)-(38), for $r = a = 1$, $s = c = \pm 1$, $g(u) = u$ and $\Omega(u) = \dfrac{2\sqrt{2}}{1 + c\,|u|^2}$.
ii) We observe that for $a = 0$, the harmonic and biharmonic functions are solutions of (33) and (34), respectively.
For $a \neq 0$, we will find solutions of (33) and (34) of the form

$$h = \langle A, B\rangle, \tag{41}$$

where A, B are holomorphic functions.
Calculating the Laplacian of (41), we get $\Delta h = 4\langle A', B'\rangle$, using this expression in (33) it follows that

$$\left\langle 1, \frac{aB}{A}\right\rangle + \left\langle \frac{2A'}{A}, \frac{B'}{A}\right\rangle = 0.$$

By Lemma 1 we obtain

$$B = -\frac{2\overline{z}_1}{a}A' + \frac{ic_1}{a}A, \tag{42}$$
$$B' = 2ic_2A' + z_1A. \tag{43}$$

From (42)

$$B' = -\frac{2\overline{z}_1}{a}A'' + \frac{ic_1}{a}A'. \tag{44}$$

Thus, from (43) and (44) we obtain

$$2\overline{z_1}A'' + i(2ac_2 - c_1)A' + az_1A = 0,$$

whose solution is given by

$$A(u) = C_1 e^{i\left(\frac{c_1-2ac_2-\Omega}{4\overline{z}_1}\right)u} + C_2 e^{i\left(\frac{c_1-2ac_2+\Omega}{4\overline{z}_1}\right)u}. \tag{45}$$

Using (45) in (42) we obtain

$$B(u) = \frac{i}{2a}\left(C_1(c_1+2ac_2+\Omega)e^{i\left(\frac{c_1-2ac_2-\Omega}{4\overline{z}_1}\right)u} + C_2(c_1+2ac_2-\Omega)e^{i\left(\frac{c_1-2ac_2+\Omega}{4\overline{z}_1}\right)u}\right). \tag{46}$$

Thus, (35) follows from (41), (45) and (46).
Similarly, calculating the Laplacian of (41) and using (34) we obtain

$$\left\langle 1, \frac{aB'}{A'}\right\rangle + \left\langle \frac{2A''}{A}, \frac{B''}{A'}\right\rangle = 0.$$

By Lemma 1 we obtain

$$B' = -\frac{2\overline{z}_1}{a}A'' + \frac{ic_1}{a}A', \tag{47}$$

$$B'' = 2ic_2A'' + z_1A'. \tag{48}$$

From (47)

$$B'' = -\frac{2\overline{z}_1}{a}A''' + \frac{ic_1}{a}A''. \tag{49}$$

Thus, from (48) and (49) we obtain

$$2\overline{z_1}A''' + i(2ac_2 - c_1)A'' + az_1A' = 0,$$

whose solution is given by

$$A(u) = -4i\overline{z_1}\left(\frac{C_1}{c_1-2ac_2-\Omega}e^{i\left(\frac{c_1-2ac_2-\Omega}{4\overline{z_1}}\right)u} + \frac{C_2}{c_1-2ac_2+\Omega}e^{i\left(\frac{c_1-2ac_2+\Omega}{4\overline{z_1}}\right)u}\right) + C_3. \tag{50}$$

Using (50) in (47) and integrating, we obtain

$$\begin{aligned} B(u) = & -\frac{i}{2a^2z_1}\left(C_1(4a|z_1|^2 - 2ac_1c_2 + c_1(c_1+\Omega))e^{i\left(\frac{c_1-2ac_2-\Omega}{4\overline{z}_1}\right)u}\right. \\ & \left. + C_2(4a|z_1|^2 - 2ac_1c_2 + c_1(c_1-\Omega))e^{i\left(\frac{c_1-2ac_2+\Omega}{4\overline{z}_1}\right)u}\right). \end{aligned} \tag{51}$$

Thus, (36) follows from (41), (50) and (51). Therefore, (3) and (4) are proven. The proof is complete. □

The following result classifies the DRMC-hypersurfaces of rotation.

Corollary 3. *Let Σ be a rotation spherical hypersurface of $\overline{M}^{n+1}(c)$ given by Theorem 3. Σ is a DRMC-hypersurface if and only if h is given by*

(1) *for $a = 0, c = 0$,*

$$h(u) = \begin{cases} C_1 + 2C_2 \ln|u|, & \text{if } n = 2, \\ \dfrac{2C_1|u|^{2-n}}{2-n} + c_2, & \text{if } n \neq 2, \end{cases}$$

(2) *for $a = 0, c = \pm 1$,*

$$h(u) = \begin{cases} C_1 + 2C_2 \ln|u|, & \text{if } n = 2, \\ C_1\left(|u|^2 - \dfrac{1}{|u|^2}\right) + 4cC_1 \ln|u| + C_2, & \text{if } n = 4, \\ C_1(-c)^{\frac{n-2}{2}} Beta\left(-c|u|^2, \frac{2-n}{2}, n-1\right) + C_2, & \text{if } n \neq 2, n \neq 4, \end{cases}$$

(3) *for $a \neq 0, c = 0$,*

$$h(u) = |u|^{1-\frac{n}{2}}\left(C_1 BesselJ\left(\frac{n}{2} - 1, \sqrt{an}|u|\right) + C_2 BesselY\left(\frac{n}{2} - 1, \sqrt{an}|u|\right)\right),$$

(4) *for $a \neq 0$, $c = \pm 1, n = 2$,*

$$\begin{aligned} h(u) \;=\; & C_1(1 + c|u|^2)^{\frac{1-\sqrt{8ac+1}}{2}} HgF_1\left(\frac{1-\sqrt{8ac+1)}}{2}, \frac{1-\sqrt{8ac+1)}}{2},\right. \\ & \left. 1-\sqrt{8ac+1}, 1 + c|u|^2\right) + C_2(1 + c|u|^2)^{\frac{1+\sqrt{8ac+1}}{2}} HgF_1\left(\frac{1+\sqrt{8ac+1}}{2},\right. \\ & \left. \frac{1+\sqrt{8ac+1}}{2}, 1+\sqrt{8ac+1}, 1 + c|u|^2\right), \end{aligned}$$

(5) *for* $a \neq 0, c = 1, n = 4,$

$$\begin{aligned} h(u) &= C_1(1+|u|^2)^{\frac{3-\sqrt{9+16a}}{2}} HgF_1\left(\frac{3-\sqrt{9+16a}}{2}, \frac{1-\sqrt{9+16a}}{2},\right. \\ &\left. 1-\sqrt{9+16a}, 1+|u|^2\right) + C_2(1+|u|^2)^{\frac{3+\sqrt{9+16a}}{2}} HgF_1\left(\frac{1+\sqrt{9+16a}}{2},\right. \\ &\left. \frac{3+\sqrt{9+16a}}{2}, 1+\sqrt{9+16a}, 1+|u|^2\right), \end{aligned}$$

(6) *for* $a \neq 0, c = -1, n = 4,$

$$\begin{aligned} h(u) &= C_1(1-|u|^2)^{\frac{3-\sqrt{9-16a}}{2}} HgF_1\left(\frac{3-\sqrt{9-16a}}{2}, \frac{1-\sqrt{9-16a}}{2},\right. \\ &\left. 1-\sqrt{9-16a}, 1-|u|^2\right) + C_2(1-|u|^2)^{\frac{3+\sqrt{9-16a}}{2}} HgF_1\left(\frac{3+\sqrt{9-16a}}{2},\right. \\ &\left. \frac{1+\sqrt{9-16a}}{2}, 1+\sqrt{9-16a}, 1-|u|^2\right), \end{aligned}$$

(7) *for* $a \neq 0, c = \pm 1, n \neq 2, n \neq 4,$

$$\begin{aligned} h(u) &= (1+c|u|^2)^{\frac{n-1-\sqrt{(n-1)^2+4acn}}{2}} \left(C_1 HgF_1\left(\frac{1-\sqrt{(n-1)^2+4acn}}{2},\right.\right. \\ &\left. \frac{n-1-\sqrt{(n-1)^2+4acn}}{2}, \frac{n}{2}, -c|u|^2\right) + C_2|u|^{2-n} \times \\ &HgF_1\left(\frac{1-\sqrt{(n-1)^2+4acn}}{2}, \frac{3-n-\sqrt{(n-1)^2+4acn}}{2},\right. \\ &\left.\left. \frac{4-n}{2}, -c|u|^2\right)\right), \end{aligned}$$

where $HgF_1 = Hypergeometric2F_1$.

Proof. From Theorem 4.17 in [10], we get that for $n \geq 2$, h is a radial function i.e. $h(u) = f(t),\ t = |u|^2$.

Differentiating the functions h and J_c, we obtain

$$\Delta h = 4tf''(t) + 2nf'(t),\ \nabla h = 2uf'(t),\ \nabla J_c = \begin{cases} 0, \text{ if } c = 0, \\ -\dfrac{16cu}{(1+ct)^3}, \text{ if } c = \pm 1. \end{cases}$$

Using these expressions in (31) we obtain

$$4tf''(t) + 2nf'(t) + anf(t) = 0, \quad for \ \ c = 0, \tag{52}$$
$$2tf''(t) + \frac{(n + ct(4-n))f'(t)}{1+ct} + \frac{2anf(t)}{(1+ct)^2} = 0, \quad for \ \ c = \pm 1. \tag{53}$$

Now we will find the solutions of equations (52) and (53).
Case: $a = 0$.
The solutions of (52) are given by
for $n = 2$

$$f(t) = C_1 + C_2 \ln t,$$

for $n \neq 2$

$$f(t) = \frac{2C_1 t^{\frac{2-n}{2}}}{2-n} + C_2.$$

The solutions of (53) are given by
for $n = 2$

$$f(t) = C_1 + C_2 \ln t,$$

for $n = 4$

$$f(t) = C_1 \left(t - \frac{1}{t}\right) + 2cC_1 \ln t + C_2,$$

for $n \neq 2, n \neq 4$

$$f(t) = C_1(-c)^{\frac{n-2}{2}} Beta\left(-ct, \frac{2-n}{2}, n-1\right) + C_2.$$

Case: $a \neq 0$.
The solutions of (52) are given by

$$f(t) = t^{\frac{1}{2}-\frac{n}{4}} \left(C_1 BesselJ\left(\frac{n}{2} - 1, \sqrt{ant}\right) + C_2 BesselY\left(\frac{n}{2} - 1, \sqrt{ant}\right)\right).$$

The solutions of (53) are given by

for $n = 2$

$$\begin{aligned} f(t) &= C_1(1+ct)^{\frac{1-\sqrt{8ac+1}}{2}} HgF_1\left(\frac{1-\sqrt{8ac+1)}}{2}, \frac{1-\sqrt{8ac+1)}}{2},\right. \\ &\left. 1-\sqrt{8ac+1}, 1+ct\right) + C_2(1+ct)^{\frac{1+\sqrt{8ac+1}}{2}} \times \\ &HgF_1\left(\frac{1+\sqrt{8ac+1}}{2}, \frac{1+\sqrt{8ac+1}}{2}, 1+\sqrt{8ac+1}, 1+ct\right), \end{aligned}$$

for $n = 4, c = 1$

$$\begin{aligned} f(t) &= C_1(1+t)^{\frac{3-\sqrt{9+16a}}{2}} HgF_1\left(\frac{3-\sqrt{9+16a}}{2}, \frac{1-\sqrt{9+16a}}{2},\right. \\ &\left. 1-\sqrt{9+16a}, 1+t\right) + C_2(1+t)^{\frac{3+\sqrt{9+16a}}{2}} \times \\ &HgF_1\left(\frac{1+\sqrt{9+16a}}{2}, \frac{3+\sqrt{9+16a}}{2}, 1+\sqrt{9+16a}, 1+t\right), \end{aligned}$$

for $n = 4, c = -1$

$$\begin{aligned} f(t) &= C_1(1-t)^{\frac{3-\sqrt{9-16a}}{2}} HgF_1\left(\frac{3-\sqrt{9-16a}}{2}, \frac{1-\sqrt{9-16a}}{2},\right. \\ &\left. 1-\sqrt{9-16a}, 1-t\right) + C_2(1-t)^{\frac{3+\sqrt{9-16a}}{2}} \times \\ &HgF_1\left(\frac{3+\sqrt{9-16a}}{2}, \frac{1+\sqrt{9-16a}}{2}, 1+\sqrt{9-16a}, 1-t\right), \end{aligned}$$

for $n \neq 2, n \neq 4$

$$\begin{aligned} f(t) &= (1+ct)^{\frac{n-1-\sqrt{(n-1)^2+4acn}}{2}} \left(C_1 HgF_1\left(\frac{1-\sqrt{(n-1)^2+4acn}}{2},\right.\right. \\ &\left. \frac{n-1-\sqrt{(n-1)^2+4acn}}{2}, \frac{n}{2}, -ct\right) + C_2 t^{\frac{2-n}{2}} \times \\ &HgF_1\left(\frac{1-\sqrt{(n-1)^2+4acn}}{2}, \frac{3-n-\sqrt{(n-1)^2+4acn}}{2},\right. \\ &\left.\left. \frac{4-n}{2}, -ct\right)\right). \end{aligned}$$

The proof is complete. □

Remark 3. From Theorem 3, in the case of HDRMC-hypersurfaces of rotation, the equation (32) is equivalent to

$$8t^2 f^{(4)}(t) + 8(n+2)tf'''(t) + 2n(n+2+at)f''(t) + an^2 f(t) = 0, \text{ if } c = 0, \quad (54)$$

$$\begin{aligned} &4t^2(1+ct)^2 f^{(4)}(t) + \left(32t^3 + 4ct^2(n+10) + 4t(n+2)\right) f'''(t) \quad (55)\\ &+ \left((56+2n-n^2)t^2 + 4(8c+an+3cn)t + n^2 + 2n\right) f''(t)\\ &+ \left(2(4-n)(n+2)t + 2an^2 + 4cn\right) f'(t) + \frac{8a(1-n)tf(t)}{(1+ct)^2} = 0, \text{ if } c = \pm 1. \end{aligned}$$

The following result classifies the HDRMC-hypersurfaces of rotation for $c = 0$ i.e. when $\overline{M}^{n+1}(0) = \mathbb{R}^{n+1}$.

Corollary 4. *Let Σ be a rotation spherical hypersurface of $\overline{M}^{n+1}(0)$ given by Theorem 3. Σ is a HDRMC-hypersurface if and only if h is given by*

(1) *for $a = 0$,*

$$h(u) = \begin{cases} (C_4 - C_2)|u|^2 + 2(|u|^2 C_2 - C_1)\ln|u| + C_3, \text{ if } n = 2,\\ C_4|u|^2 - \dfrac{(3C_2 - \sqrt{15}C_1)\cos\left(\sqrt{15}\ln|u|\right) + (\sqrt{15}C_2 + 3C_1)\sin\left(\sqrt{15}\ln|u|\right)}{24|u|}\\ +C_3, \text{ if } n = 4,\\ \dfrac{4|u|^{2-n}\left((n-4)C_1 + nC_2|u|^2\right)}{n(n-2)(n-4)} + C_4|u|^2 + C_3, \text{ if } n \neq 2, n \neq 4, \end{cases}$$

(2) *for $a \neq 0$,*

$$h(u)=\begin{cases}\dfrac{2aC_1\ln|u|-2C_2 BesselJ\left(0,\sqrt{2a}|u|\right)-4C_3 BesselY\left(0,\sqrt{2a}|u|\right)+2C_2}{a}\\ +C_4,\ \ \textit{if}\ n=2,\\ -2C_3 MeijerG\left(\{\{0\},\{-\frac{1}{2}\}\},\{\{0,0\},\{-1,-\frac{1}{2}\}\},\sqrt{a}|u|,\frac{1}{2}\right)\\ -\dfrac{C_2 BesselI\left(1,2\sqrt{-a}|u|\right)}{\sqrt{-a}|u|}-\dfrac{C_1}{|u|^2}+C_2,\ \ \textit{if}\ n=4,\\ \dfrac{|u|^{-n}}{a^2n}\left(\dfrac{2^{2-n}(an|u|^2)^{\frac{n}{4}}C_3 Gamma\left(\frac{n}{2}\right)}{Gamma\left(\frac{n+2}{2}\right)}\left(2(an|u|^2)^{\frac{n}{4}}-2^{\frac{n}{2}}\sqrt{an}|u|\times\right.\right.\\ \left. BesselJ\left(\frac{n-2}{2},\sqrt{an}|u|\right)Gamma\left(\frac{n}{2}\right)\right)-\dfrac{a|u|^2}{n-2}\left(2anC_1-4C_2(n-2)\right.\\ \left.\left.+C_2n(n-2)^2 Gamma\left(-\frac{n}{2}\right)HgF_1R\left(\frac{4-n}{2},-\frac{an|u|^2}{4}\right)\right)\right)+C_4,\\ \textit{if}\ n\neq 2, n\neq 4,\end{cases}$$

where $HgF_1R = Hypergeometric0F1Regularized.$

Proof. Similarly to the proof of Corollary 3, from Theorem 4.17 in [10], we get that for $n \geq 2$, h is a radial function i.e. $h(u) = f(t)$, $t = |u|^2$.
On the other hand, from Remark 3 the expression (32) is equivalent to (54), thus, we will find the solutions of this equation.
The solutions of equation (54) are given by
for $a = 0$

$$f(t)=\begin{cases}(C_4-C_2)t+(tC_2-C_1)\ln t+C_3,\ \text{if}\ n=2,\\ C_4t-\dfrac{(3C_2-\sqrt{15}C_1)\cos\left(\frac{\sqrt{15}\ln t}{2}\right)+(\sqrt{15}C_2+3C_1)\sin\left(\frac{\sqrt{15}\ln t}{2}\right)}{24\sqrt{t}}\\ +C_3,\ \text{if}\ n=4,\\ \dfrac{4t^{\frac{2-n}{2}}\left((n-4)C_1+nC_2t\right)}{n(n-2)(n-4)}+C_4t+C_3, \text{if}\ n\neq 2, n\neq 4,\end{cases}$$

for $a \neq 0$

$$f(t) = \begin{cases} \dfrac{aC_1 \ln t - 2C_2 BesselJ\left(0, \sqrt{2at}\right) - 4C_3 BesselY\left(0, \sqrt{2at}\right) + 2C_2}{a} \\ +C_4, \text{ if } n = 2, \\ -2C_3 MeijerG\left(\{\{0\}, \{-\frac{1}{2}\}\}, \{\{0,0\}, \{-1, -\frac{1}{2}\}\}, \sqrt{at}, \frac{1}{2}\right) \\ -\frac{C_2 BesselI\left(1, 2\sqrt{-at}\right)}{\sqrt{-at}} - \frac{C_1}{t} + C_2, \text{ if } n = 4, \\ \dfrac{t^{-\frac{n}{2}}}{a^2 n}\left(\dfrac{2^{2-n}(ant)^{\frac{n}{4}} C_3 Gamma\left(\frac{n}{2}\right)}{Gamma\left(\frac{n+2}{2}\right)}\left(2(ant)^{\frac{n}{4}} - 2^{\frac{n}{2}}\sqrt{ant}\times\right.\right. \\ \left. BesselJ\left(\frac{n-2}{2}, \sqrt{ant}\right) Gamma\left(\frac{n}{2}\right)\right) - \frac{at}{n-2}\left(2anC_1 - 4C_2(n-2)\right. \\ \left.\left. +C_2 n(n-2)^2 Gamma\left(-\frac{n}{2}\right) HgF_1R\left(\frac{4-n}{2}, -\frac{ant}{4}\right)\right)\right) + C_4, \\ \text{if } n \neq 2, n \neq 4. \end{cases}$$

The proof is complete. □

4 Conclusions

The DRMC-hypersurfaces and the HDRMC-hypersurfaces in space forms $\overline{M}^{n+1}(c)$, $c = -1, 0, 1$ generalize the Weingarten hypersurfaces of the spherical type studied by [10]. In the case $n = 2$, using two holomorphic functions a way to construct DRMC-surfaces and HDRMC-surfaces in $\overline{M}^3(c)$ is obtained. Finally, as a first step, we classify the DRMC-hypersurfaces of rotation in $\overline{M}^{n+1}(c)$ and the HDRMC-hypersurfaces of rotation in $\mathbb{R}^{n+1}$. It would be interesting to study DRMC-hypersurfaces and HDRMC-hypersurfaces with some geometric properties such as embeddededness, completeness. In this sense, future research is being carried out.

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