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On Spherically Symmetric Douglas Metrics with Vanishing S-curvature with Explicit Examples

Sobre Métricas de Douglas Esfericamente Simétricas com S-curvatura Nula e Exemplos

Newton Solórzano*
Marcelo Souza†
Edwin López‡

Abstract: We obtain the differential equations that characterize the spherically symmetric Douglas metrics with vanishing S-curvature. We study these equations and obtain conditions on such metrics. Many explicit examples are included.

Keywords: Finsler metric. Douglas curvature. S-curvature. Spherical symmetry.

Resumo: Obtemos as equações diferenciais que caracterizam as métricas Douglas esfericamente simétricas com S-curvatura nula. Estudamos essas equações e obtemos condições sobre tais métricas. Muitos exemplos explícitos estão incluídos.

Palavras-chave: Métrica de Finsler. Curvatura de Douglas. S-curvatura. Simetria esférica.

1 Introduction

Z. Shen introduced the notion of S-curvature of a Finsler space in [11]. It is a quantity to measure the rate of change of the volume form of a Finsler space along the geodesics. The S-curvature is a non-Riemannian quantity, i.e., any Riemannian manifold has the S-curvature vanishing everywhere. It is well known that, for a Finsler metric F of scalar flag curvature, if the S-curvature is almost isotropic, i.e.,

$$S = (n-1)cF + \eta,$$

*Universidade Federal da Integração Latino-Americana, newton.chavez@unila.edu.br
†Instituto de Matemática e Estatística. Universidade Federal de Goiás - UFG, msouza@ufg.br
‡Universidade Federal de Roraima, edwin.lopez@ufrr.br

where $c = c(x)$ is a scalar function and η is any closed 1−form, then the flag curvature must be in the following form

$$K = \frac{3\bar{c}_{x_m} y^m}{F} + \sigma,$$

where $\sigma = \sigma(x)$ and $\bar{c} = \bar{c}(x)$ are scalar functions with $c - \bar{c} = constant$ [1].

A Finsler metric on a manifold M is a *Douglas metric* if its Douglas curvature vanishes. The Douglas curvature was introduced by J. Douglas in 1927 [2]. Its importance in Finsler geometry is due to the fact that it is a projective invariant quantity. Namely, if two Finsler metrics F and $\bar{F}$ are projectively equivalent, then F and $\bar{F}$ have the same Douglas curvature. The class of Douglas metrics contains all Riemannian metrics and the locally projectively flat Finsler metrics.

On the other hand, let M^n_s be the open ball $\mathbf{B}^n(\nu) := \{x \in \mathbb{R}^n : |x| < \nu\}$, or the annuli domain $\mathbf{B}(\nu_1) \setminus \mathbf{B}(\nu_2)$, both centered at the origin, or the euclidean space $\mathbb{R}^n$, where $\infty > \nu > 0$ and $\nu_1 > \nu_2 \geq 0$. $|\cdot|$ is the standard Euclidean norm.

Finsler metrics defined on M^n_s satisfying

$$F(Ax, Ay) = F(x, y), \tag{1}$$

for all $A \in O(n)$, are called *spherically symmetric* (*orthogonally invariant* in an alternative terminology in [10]). Such metrics were first studied by Rutz in [7].

In [6] it was obtained all the spherically symmetric Douglas metrics on a symmetric subspace of $\mathbb{R}^n$, and we can see that there are a lot of them.

In this paper, we first characterize Douglas metrics with vanishing S-curvature and Berwald metrics, in terms of a differential equation System (Theorem 1 and Theorem 2). Then comparing these systems we conclude that they are equivalent, and finally we obtain all the spherically symmetric Douglas metrics with vanishing S-curvature (Theorem 3).

From examples of Douglas metrics, we obtain new examples of Berwald metrics (see Section 6, below $|\cdot|$ and $\langle\cdot,\cdot\rangle$ are the standard Euclidean norm and inner product on $\mathbb{R}^n$):

$$F(x, y) = c|y|e^{-\int_0^{|x|} 2rg dr} \tag{2}$$

$$F(x, y) = c_1 \frac{\langle x, y\rangle}{|x|^2} + c_2 \frac{|y|}{|x|} \tag{3}$$

see Corollary 1, which is more treatable than the second case of Theorem 3, it provides this notable non-Riemannian Berwald metric

$$F(x,y) = \left(c_1\frac{\langle x,y\rangle}{|x|^2} + c_2\frac{|y|}{|x|}\right)\left(e^{\frac{|x|^2|y|^2-\langle x,y\rangle^2}{|x|^2|y|^2}} + 5\right). \tag{4}$$

It is convenient to mention that in [14] the author characterized Douglas metrics with isotropic S-curvature. Unlike this article, we gave alternative techniques to the one used in [14] and additionally we presented several non evident examples of Douglas metrics with vanish S-curvature. Then, the spirit of this paper is to give a new tool for the better understanding of spherically symmetric Finsler metrics.

Firstly we introduce the notation

$$r := |x|, \qquad s := \frac{\langle x,\, y\rangle}{|y|}, \tag{5}$$

where $|\cdot|$ and $\langle\cdot,\cdot\rangle$ are the standard Euclidean norm and inner product on $\mathbb{R}^n$. In Section 4, we prove the following Theorems.

Theorem 1. *On M_s^n, a spherically symmetric Finsler metric $F(x,\, y) = |y|\phi(r,\, s)$ is of Douglas type with vanishing S-curvature if, and only if, ϕ satisfies*

$$\left[(r^2-s^2)(2g+fs^2)-1\right]r\phi_{ss} - s\phi_{rs} + \phi_r + r(2g+fs^2)(\phi - s\phi_s) = 0, \tag{6}$$

$$\left[1-(r^2-s^2)(2g+fs^2)\right]r\phi_s + s\phi_r - rs(2g+s^2f)\phi - s\left[\frac{2}{n+1}\left(\frac{A'_\phi(r)}{A_\phi(r)} - r(2g+r^2f)\right)\right]\phi = 0, \tag{7}$$

where r and s are defined in (5), $f = f(r)$ and $g = g(r)$ are some differentiable functions, and $A_\phi(|x|)dx$ is the volume form of F.

There are two important volume forms in Finsler geometry. One is the Busemann-Hausdorff volume form and the other is the Holmes-Thompson volume form.

Theorem 2. *On M_s^n, a spherically symmetric Finsler metric $F(x,y) = |y|\phi(r,s)$ is Berwald type if, and only if, ϕ satisfies:*

$$\left[(r^2-s^2)(2g+fs^2)-1\right]r\phi_{ss} - s\phi_{rs} + \phi_r + r(2g+fs^2)(\phi - s\phi_s) = 0, \tag{8}$$

$$\left[1-(r^2-s^2)(2g+fs^2)\right]r\phi_s + s\phi_r - sr\left(2g+s^2f\right)\phi - 2rsL(r)\phi = 0, \tag{9}$$

where r and s are defined in (5), $f = f(r)$, $g = g(r)$ and $L(r)$ are some differentiable functions.

By using the characteristic curves, our next result provides how is the form of ϕ, which is solution of (6) and (7).

Let $f(r)$ and $g(r)$ be differentiable functions, for convenience we define $I = I(r)$ and $II = II(r)$ for $r < \nu$ as:

$$I = \int 2r(2g + r^2 f)dr \qquad \text{and} \qquad II = \int 2rfe^{\int 2r(2g+r^2 f)dr}dr \tag{10}$$

and additionally we suppose that

$$(r^2 - s^2)II(r) - e^{I(r)} \neq 0, \quad \forall (r,s) \in [0,\nu) \times (-\nu,\nu). \tag{11}$$

In Section 5, we prove the next theorem.

Theorem 3. *Let $f(r)$ and $g(r)$ be differentiable functions of $r \in J \subset \mathbf{R}$ such that I and II in (10) are well defined and (11) is satisfied. Suppose $F(x,y) = |y|\phi(r,s)$ is a Douglas type with vanish S-curvature (with respect to Busemann-Hausdorff volume form), then, up to homothety, we have:*

1. *If $g \neq \frac{1}{2r^2}$,*

$$F(x,y) = \frac{\sqrt{\left|\left(|x|^2|y|^2 - \langle x,y\rangle^2\right)\int 2rfe^{\int 2r(2g+r^2 f)dr}dr - e^{\int 2r(2g+r^2f)dr}|y|^2\right|}}{\left|r^2\int 2rfe^{\int 2r(2g+r^2f)dr}dr - e^{\int 2r(2g+r^2f)dr}\right|}, \tag{12}$$

2. *If $g = \frac{1}{2r^2}$,*

$$F(x,y) = \frac{\sqrt{|x|^2|y|^2 - \langle x,y\rangle^2}}{|x|^2}\eta\left(\frac{|x|^2|y|^2 - \langle x,y\rangle^2}{\langle x,y\rangle^2}e^{-\int 2r^3 f(r)dr}\right), \tag{13}$$

where η is any differentiable function, such that:

$$3\frac{\sqrt{r^2-s^2}}{s^4}\eta' + 2\frac{(r^2-s^2)^{3/2}}{s^6}e^{-\int 2r^3 fdr} \quad \eta'' > 0 \ \text{when}\ n \geq 2, \tag{14}$$

with the additional condition

$$\frac{\eta}{\sqrt{r^2-s^2}} + \frac{2\sqrt{r^2-s^2}}{s^2}e^{-\int 2r^3 fdr} \quad \eta' > 0 \ \text{when}\ n \geq 3. \tag{15}$$

By using the Formula (13) and inspired by Example 3, we have,

Corollary 1. *Let $\phi(r,s)$ defined by:*

$$\phi(r,s)=\left(c_1\frac{s}{r^2}+c_2\frac{1}{r}\right)\overline{\eta}(\varphi_1(r,s),\varphi_2(r,s)),$$

where c_1 and c_2 are real constants and $\overline{\eta}$ is any smooth function of $\varphi_1(r,s)=\frac{r}{\sqrt{r^2-s^2}}$ and $\varphi_2(r,s)=\frac{s}{\sqrt{r^2-s^2}}$. Then $\phi(r,s)$ satisfies Equations (8) and (9) with $f(r)=0$.

Remark 1. *Examples 3 and 4 are type (13).*

L. Zhou [14], showed that every spherically symmetric Landsberg metric is of Berwald type.

By definition, S-curvature $S(y)$ measures the average rate of changes of (T_xM, F_x) in the direction of $y\in T_xM$. An important property is that $S=0$ for Berwald metrics with respect to the Busemann-Hausdorff volume form dV_{BH} (see [11], [8]). Then, from Theorem 1 and Theorem 2 we conclude that every spherically symmetric Douglas metric with vanishing S-curvature are Berwald (Landsberg) metric.

Theorem 4. *Let $F=|y|\phi(r,s)$ be a spherically symmetric Finsler metric, then it is of Douglas type with vanish S-curvature on M_s^n if, and only if, it is of Landsberg type, and ϕ is characterized by Theorem 3.*

Corollary 2. *Let $F=|y|\phi(r,s)$ be a spherically symmetric Finsler metric defined on $\mathbf{R}^n$, then F is Berwald type if, and only if, it is of Riemannian type given by (12).*

The next corollary surprisingly characterizes the Busemann-Hausdorff volume form of Berwald metrics in the second case of Theorem 3.

Corollary 3. *Let $F=|y|\phi(r,s)$ be a Berwald metric of (13) type, then the Busemann-Hausdorff element of volume, $A_\phi(r)$, up to homothety, is given by*

$$A_\phi(r)=\frac{e^{\int r^3 f dr}}{r^n}.$$

Remark 2. *When the volume form is of Holmes-Thompson type, the characterization Theorem 3 can be rewritten by adding in the Item (ii) the condition about $f(r)$ and $\eta(\varphi)$:*

$$A_\phi(r)=\frac{ce^{\int r^3 f dr}}{r^n},$$

where $A_\phi(r)dx$ is the Holmes-Thompson volume form given in the Proposition 1, and c is a positive real constant.

2 Preliminaries

In this section, we give some definitions and lemmas that will be used in the proof of our main results.

Let M be a manifold and let $TM = \cup_{x\in M} T_xM$ be the tangent bundle of M, where T_xM is the tangent space at $x \in M$. We set $TM_o := TM \setminus \{0\}$, where $\{0\}$ stands for $\{(x,\, 0) |\, x \in M,\, 0 \in T_xM\}$. A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ with the following properties

1. F is C^∞ on TM_o;
2. At each point $x \in M$, the restriction $F_x := F|_{T_xM}$ is a Minkowski norm on T_xM.

Let $F = F(x, y)$ be a Finsler metric on an $n-$dimensional manifold M. Let $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ denote the spray of F. The spray coefficients G^i are defined by

$$G^i = \frac{1}{4} g^{il} \left\{ [F^2]_{x^j y^l} y^j - [F^2]_{x^l} \right\}.$$

A Finsler metric F on $M^n(\nu)$ is said to be *spherically symmetric* if it satisfies (1) for all $x \in \mathbf{B}^n(\nu)$, $y \in T_x\mathbf{B}^n(\nu)$ and $A \in O(n)$. In [13], Zhou (see also [4]) showed the following:

Lemma 1. *[4] A Finsler metric F on $\mathbf{B}^n(\nu)$ is spherically symmetric if, and only if, there is a function $\phi : [0,\, \nu) \times \mathbb{R} \to \mathbb{R}$ such that*

$$F(x, y) = |y| \phi \left(|x|, \frac{\langle x,\, y\rangle}{|y|} \right),$$

where $(x, y) \in \mathcal{T}\mathbf{B}^n(\nu) := T\mathbf{B}^n(\nu) \setminus \{0\}$.

Note that spherically symmetric Finsler metrics are general (α, β)-metrics studied in [9].

A Finsler metric on a manifold M is called a *Douglas metric* if its geodesic coefficients $G^i = G^i(x,\, y)$ are in the following form

$$G^i = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k + P(x,\, y) y^i,$$

where $\Gamma^i_{jk}(x)$ are functions on M, in local coordinates, and $P(x,\, y)$ is a local positively y-homogeneous function of degree one, if $P = 0$ then the metric F is called Berwald metric.

A straightforward computation shows the following result that was proved independently in [3], [5] and [12].

Lemma 2. *Let* $F = |y|\phi\left(|x|, \frac{\langle x,y\rangle}{|y|}\right)$ *be a spherically symmetric Finsler metric on* $\mathbf{B}^n(\nu) \subset \mathbf{R}^n$. *Let* $x^1, \cdots, x^n$ *be the coordinates on* $\mathbf{R}^n$ *and denote by* $y = \sum y^i \partial/\partial x_i$. *Then its geodesic coefficients are given by*

$$G^i = |y|Py^i + |y|^2 Qx^i, \tag{16}$$

where

$$Q := \frac{1}{2r}\frac{r\phi_{ss} - \phi_r + s\phi_{rs}}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}}, \qquad r := |x|, \quad s := \frac{\langle x, y\rangle}{|y|} \tag{17}$$

and

$$P := \frac{r\phi_s + s\phi_r}{2r\phi} - \frac{Q}{\phi}\left[s\phi + (r^2 - s^2)\phi_s\right].$$

The Busemann-Hausdorff volume form $dV_{BH} = \sigma_{BH}(x)dx$ is given by

$$\sigma_{BH}(x) = \frac{\omega_n}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n | F(x, y^i \frac{\partial}{\partial x^i}) < 1\}}$$

and the Holmes-Thompson volume form $dV_{HT}(x) = \sigma_{HT}(x)dx$ is given by

$$\sigma_{HT}(x) = \frac{1}{\omega_n}\int_{\{(y^i)\in\mathbf{R}^n | F(x, y^i \frac{\partial}{\partial x^i})<1\}} det(g_{ij})dy.$$

Here Vol denotes the Euclidean volume and

$$\omega_n := \mathrm{Vol}(B^n(1)) = \frac{1}{n}\mathrm{Vol}(S^{n-1}) = \frac{1}{n}\mathrm{Vol}(S^{n-2})\int_0^\pi sin^{n-2}(t)dt$$

denotes the Euclidean volume of the unit ball in $\mathbb{R}^n$.

For a general (α, β)-metric we have the following formulas for the volume forms dV_{BH} and dV_{HT}.

Proposition 1. *Let* $F = \alpha\phi(\beta, \beta/\alpha)$ *be a general* (α, β) *metric on an* n*-dimensional* *manifold M. Let dV be or dV_{BH} or dV_{HT}. Let*

$$A(b) := \begin{cases} \dfrac{\displaystyle\int_0^\pi \sin^{n-2}(t)dt}{\displaystyle\int_0^\pi \frac{\sin^{n-2}(t)}{\phi(b, bcos(t))^n}dt} & \text{if } dV = dV_{BH} \\ \\ \dfrac{\displaystyle\int_0^\pi (\sin^{n-2}(t))T(b\cos(t))dt}{\displaystyle\int_0^\pi \sin^{n-2}(t)dt} & \text{if } dV = dV_{HT}, \end{cases} \tag{18}$$

where $b = ||\beta||_\alpha$ and $T(s) := \phi(\phi - s\phi_2)^{n-2}[(\phi - s\phi_2) + (b^2 - s^2)\phi_{22}]$. Then the volume form dV is given by

$$dV = A(b)dV_\alpha,$$

where $dV_\alpha = \sqrt{det(a_{ij})}dx$ denotes the Riemannian volume form of α.

The next result is given in [6].

Lemma 3. *[6] Given $f(r)$ and $g(r)$ differentiable functions, such that I and II in (10) are well defined, then*

$$\left[1 - (r^2 - s^2)(2g + s^2 f)\right] r\psi_s(r, s) + s\psi_r(r, s) = 0 \tag{19}$$

is equivalent to

$$\psi(r, s) = \eta(\varphi(r, s))), \tag{20}$$

where η is any differentiable real function of

$$\varphi(r, s) = \frac{r^2 - s^2}{(r^2 - s^2)\int 2rfe^{\int 2r(2g+r^2 f)dr}dr - e^{\int 2r(2g+r^2 f)dr}}. \tag{21}$$

A variation of the last lemma:

Proposition 2. *Let $f = f(r)$, $g = g(r)$ and $L(r)$ be differentiable functions of $r \in (0, r_0)$ such that $L(r)$ is integrable and conditions (10) hold. Then, for $r \neq s$, a positive function $\phi(r, s)$ defined on $(0, r_0) \times \mathbb{R}$ satisfies*

$$[1 - (r^2 - s^2)(2g + s^2 f)]r\phi_s(r, s) + s\phi_r(r, s) - s[r(2g + s^2 f) + L(r)]\phi(r, s) = 0, \tag{22}$$

if, and only if,

$$\phi(r, s) = e^{\int L(r)dr}\sqrt{r^2 - s^2}\eta(\varphi),$$

where η is any positive function of φ defined in (21).

Proof. Observe that Equation (22) is equivalent to next transport equation:

$$[1-(r^2-s^2)(2g+s^2f)]r\psi_s(r,s)+s\psi_r(r,s)=0,$$

where

$$\psi(r,s)=\frac{e^{-\int L_\phi(r)dr}\phi(r,s)}{\sqrt{r^2-s^2}}.$$

□

3 Douglas curvature, S−curvature and Berwald curvature of spherically symmetric Finsler metrics

In [2], Douglas introduced the local functions ${D_j}^i{}_{kl}$ on $\mathcal{T}\mathbf{B}^n(\nu)$ defined by

$$ {D_j}^i{}_{kl}:=\frac{\partial^3}{\partial y^j\partial y^k\partial y^l}\left(G^i-\frac{1}{n+1}\sum_m\frac{\partial G^m}{\partial y^m}y^i\right), \tag{23}$$

in local coordinates $x^1,\cdots,x^n$ and $y=\sum_i y^i\partial/\partial x^i$. These functions are called *Douglas curvature* [2] and a Finsler metric F is said to be a *Douglas metric* if ${D_j}^i{}_{kl}=0$.

A Finsler metric F is said Berwald metric if $B^i_{jkl}=0$, where

$$B^i_{jkl}=\frac{\partial^3G^i}{\partial y^j\partial y^k\partial y^l}.$$

Let $dV=\sigma(x)dx$ be the volume form of (M,F), then the S-curvature (with respect to dV) is defined by

$$S=\frac{\partial G^m}{\partial y^m}-y^m\frac{\partial}{\partial x^m}(\ln\sigma).$$

The S-curvature is associated with a volume form.

4 Spherically symmetric Douglas metrics with vanishing S-curvature

In our next result, we obtain the Douglas curvature and the S-curvature of a spherically symmetric Finsler metric on M_s^n with respect to Busemann-Hausdorff volume form.

Now, we are going to discuss necessary and sufficient conditions for a spherically symmetric metric to be Douglas type with vanishing S-curvature.

In [6], we prove the following:

Lemma 4 ([6])**.** *On M_s^n, a spherically symmetric Finsler metric $F(x,y) = |y|\phi(r,s)$ is of Douglas type if, and only if, ϕ satisfies*

$$\left[(r^2 - s^2)(2g + fs^2) - 1\right] r\phi_{ss} - s\phi_{rs} + \phi_r + r(2g + fs^2)(\phi - s\phi_s) = 0, \tag{24}$$

where r and s are defined in (5), $f = f(r)$ and $g = g(r)$ are some differentiable functions.

A direct computation gives the expression of S-curvature, presented in Lemma 5.

Lemma 5. *Let $F = |y|\phi(|x|, \frac{\langle x,y\rangle}{|y|})$ be a spherically symmetric Finsler metric on $\mathbf{B}^n(\nu) \subseteq \mathbf{R}^n$. Then the S-curvature of F is given by*

$$S = |y|[(n+1)P + 2sQ + (r^2 - s^2)Q_s] - |y|\frac{s}{r}\frac{A'(r)}{A(r)}, \tag{25}$$

where r and s are defined in (5), Q and P are defined in Lemma 2.2, and $A(r) := \sigma(x)$.

Proof of Theorem 1.1. Note that Equation (6) is equivalent to $Q = g(r) + \frac{s^2}{2}f(r)$, so we replace it in (25), and conclude the proof of Theorem 1.1. □

Lemma 6. *Let $F = |y|\phi(r,s)$ be a spherically symmetric Finsler metric on M_s^n then*

F is a Berwald metric if, and only if,

$$0 = \frac{P_{ss}}{u}\left(x^j x^k \delta^i_l\right)_{\overrightarrow{jkl}} + \frac{(P - sP_s)}{u}\left(\delta_{jk}\delta^i_l\right)_{\overrightarrow{jkl}} - \frac{sP_{ss}}{u^2}\left\{\left(\delta_{jk}x^l y^i\right)_{\overrightarrow{jkl}} + \left(\left(x^k y^j\right)_{\overrightarrow{jk}}\delta^i_l\right)_{\overrightarrow{jkl}}\right\}$$
$$+ \frac{1}{u^3}\left\{-P + sP_s + s^2 P_{ss}\right\}\left(y^j y^k \delta^i_l\right)_{\overrightarrow{jkl}} + \frac{1}{u^3}\left\{-P + sP_s + s^2 P_{ss}\right\} y^i \left(\delta_{jk} y^l\right)_{\overrightarrow{jkl}}$$
$$+ \frac{1}{u^5}\left\{3P - 3sP_s - 6s^2 P_{ss} - s^3 P_{sss}\right\} y^i y^k y^l y^i + \frac{P_{sss}}{u^2} x^j x^k x^l y^i$$
$$+ \frac{s}{u^4}\left\{3P_{ss} + sP_{sss}\right\}\left(x^j y^k y^l\right)_{\overrightarrow{jkl}} y^i + \frac{1}{u^3}\left\{-P_{ss} - sP_{sss}\right\}\left(x^j x^k y^l\right)_{\overrightarrow{jkl}} y^i.$$

Proof. The proof is analogous to the proof of the Proposition 3.1 given in [6], and remembering that every Berwald metric is a Douglas metric. □

The previous lemma is equivalent to

Lemma 7. *Let $F = |y|\phi(r,s)$ be a spherically symmetric Finsler metric on M^n_s, then F is of Berwald type if, and only if, there exist $L := L(r)$, $f := f(r)$ and $g := g(r)$ such that*

$$P = L, \quad Q = \frac{1}{2} f s^2 + g.$$

Proof of Theorem 2. Lemma 7 is equivalent to Theorem 2. □

5 Proof of Theorem 3

By Equation (9) and Proposition 2 the next identity must be satisfied:

$$\phi(r,s) = e^{\int L(r)dr}\sqrt{r^2 - s^2}\,\eta(\varphi(r,s)) \tag{26}$$

for some differentiable function η of φ. Combining (26) with (6), we obtain that the next identity must be satisfied too

$$\left[2r + r^2 L(r)\right](r^2 - s^2)\eta(\varphi) = e^{\int 2r(2g + r^2 f)dr}\left[4r - 4r2g(r^2 - s^2) + 2s^2 L(r)\right]\varphi^2 \eta'(\varphi). \tag{27}$$

If $L(r) = -\frac{2}{r}$ then $g = \frac{1}{2r^2}$, and we obtain (ii) of Theorem 3.

Observe that if $g(r) = \frac{1}{2r^2}$ and $L(r) \neq -\frac{2}{r}$ then we can rewrite (27):

$$\frac{\eta_s(\varphi(r,s))}{\eta(\varphi(r,s))} = \frac{r^2}{s(r^2 - s^2)}.$$

Integrating the last equation with respect to s we have:

$$\eta(\varphi(r,s)) = e^{c(r)}\frac{|s|}{\sqrt{r^2-s^2}},$$

where $c(r)$ is a constant of integration in s. This function does not define a Finsler metric $(\phi(r,s) - s\phi_s(r,s) = 0)$.

If $L(r) \neq -\frac{2}{r}$, then identity (27) can be rewritten as

$$\frac{\eta(\varphi)}{\varphi^2\eta'(\varphi)} = \frac{e^{\int 2r(2g+r^2f)dr}}{(r^2-s^2)(2r+r^2L(r))}(4r - 4r2g(r^2-s^2) + 2s^2L(r)). \qquad (28)$$

By Lemma 3, the right hand side of (28) should satisfy the transport equation:

$$\left[1-(r^2-s^2)(2g+s^2f)\right] r\psi_s(r,s) + s\psi_r(r,s) = 0. \qquad (29)$$

Then, a straightforward computation shows

$$r\,a(r)L_\phi'(r) - ra(r)L^2(r) + (rb(r)-a(r))L(r) + 2b(r) = 0, \qquad (30)$$

where $a(r) := 2r^2g(r) - 1$ and $b(r) := -2r^2g'(r) - 4r^3(2g(r)+r^2f(r))g(r) + 2r^3f(r)$.

If we see Equation (28) as an ODE for $L(r)$, then we obtain a necessarily condition for $L(r)$.

Observe that $-\frac{2}{r}$ is a particular solution of the Ricatti Equation (30). Consequently we obtain the general solution

$$L(r) := -\frac{2}{r} - \frac{e^{-\int \frac{b(r)}{a(r)}dr}}{r^3\left(\int \frac{1}{r^3}e^{-\int \frac{b(r)}{a(r)}dr}dr\right)}. \qquad (31)$$

Using the last identity, (28) can be rewritten as:

$$\frac{\eta_s(\varphi(r,s))}{\eta(\varphi)} = \frac{2s}{(r^2-s^2)^2}\frac{-r^2(r^2-s^2)e^{-\int \frac{b(r)}{a(r)}dr}}{4r^2a(r)(s^2-r^2)\left(\int \frac{1}{r^3}e^{-\int \frac{b(r)}{a(r)}dr}dr\right) - 2s^2e^{-\int \frac{b(r)}{a(r)}dr}}.$$

Integrating in s the last identity, we can obtain how should look like $\eta(\varphi)$:

$$\eta(\varphi(r,s)) = \frac{\left|4r^2a(r)(s^2-r^2)\left(\int \frac{1}{r^3}e^{-\int \frac{b(r)}{a(r)}dr}dr\right) - 2s^2e^{-\int \frac{b(r)}{a(r)}dr}\right|^{\frac{1}{2}}}{\sqrt{r^2-s^2}}\,T(r), \qquad (32)$$

where $T(r)$ is a constant of integration in s, and in order to the last identity makes sense (see Lemma (3)), then $T(r)$ must be

$$T(r) = C_2 \frac{1}{r|2r^2g-1|^{\frac{1}{2}}},$$

where C_2 is a positive real constant. Once the following equations are satisfied,

$$\int \frac{1}{r^3} e^{-\int \frac{b(r)}{a(r)}dr} dr = \frac{e^{-\int \frac{b(r)}{a(r)}dr}}{2r^2 a(r)} + \int \frac{-rf}{a(r)} e^{-\int \frac{b(r)}{a(r)}dr} dr$$

and

$$e^{-\int \frac{b(r)}{a(r)}dr} = |a(r)| e^{\int 2r(2g+r^2f)dr},$$

we obtain (12), up to homothety.

We observe that C. Yu and H. Zhu, [12], gave necessary and sufficient conditions for $F = \alpha\phi(\|\beta_x\|_\alpha, \frac{\beta}{\alpha})$ to be a Finsler metric for any α and β with $\|\beta_x\|_\alpha < b_0$. In particular, considering $F(x,y) = |y|\phi\left(|x|, \frac{\langle x,y\rangle}{|y|}\right)$, then F is a Finsler metric if, and only if, the positive function ϕ satisfies

$$\phi(r,s) - s\phi_s(r,s) + (r^2-s^2)\phi_{ss}(r,s) > 0, \qquad \text{when } n \geq 2,$$

with the additional inequality

$$\phi(r,s) - s\phi_s(r,s) > 0, \qquad \text{when } n \geq 3.$$

Therefore, when ϕ is given by (13), F defines a Finsler metric if, and only if, the Inequalities (14) and (15) hold. □

6 Examples

Example 6.1. Considering $f(r) = g(r) = 0$ in Theorem 3 we obtain, up to homothety, the next projectively flat metric with vanishing S-curvature:

$$F(x,y) = \frac{\sqrt{|2c_1(|x|^2|y|^2 - \langle x,y\rangle^2) - c_2|y|^2|}}{|c_2|x|^2 - c_1|}, \tag{33}$$

where $c_1 > 0$ and c_2 are any real constants such that

$$2c_1\left(|x|^2|y|^2 - \langle x,y\rangle^2\right) - c_2|y|^2 \neq 0 \quad \text{and} \quad c_2|x|^2 - c_1 \neq 0.$$

From the Douglas metric:

$$F(x,y) = \langle x,y\rangle h(|x|) + c|y|e^{-\int_0^{|x|} 2rgdr},$$

where c is any positive real constant. We obtain the next examples:

Example 6.2. Considering $h(|x|) = 0$, the Douglas metric

$$F(x,y) = c|y|e^{-\int_0^{|x|} 2rgdr}$$

has vanishing S-curvature.

Example 6.3. Considering an annuli domain $B^n(\nu_1)\backslash B^n(\nu_2)$, $(\nu_1 > \nu_2 > 0)$, and $g(r) = \frac{1}{2r^2}$, the corresponding Douglas metric

$$F(x,y) = c_1\frac{\langle x,y\rangle}{|x|^2} + c_2\frac{|y|}{|x|}$$

has vanishing S-curvature.

Example 6.4. Considering an annuli domain $B^n(\nu_1)\backslash B^n(\nu_2)$, $(\nu_1 > \nu_2 > 0)$, $g(r) = \frac{1}{2r^2}$ and $\overline{\eta}(\varphi_1,\varphi_2) = e^{\varphi_1^{-2}} + 5$, in the Corollary 1, we have that the next Douglas metric

$$F(x,y) = \left(\frac{\langle x,y\rangle}{|x|^2} + \frac{|y|}{|x|}\right)\left(e^{\frac{|x|^2|y|^2 - <x,y>^2}{|x|^2|y|^2}} + 5\right)$$

has vanishing S-curvature.

Remark 3. *An interesting consequence of the Example 6.1 is that on* $\mathbf{R}^n$*,* $F(x,y)$ *is projectively flat metric with vanishing* S*-curvature if, and only if, up homothety,*

$$F(x,y) = |y|.$$

References

[1] CHEN, X.; MO, X.; SHEN, Z. On the Flag Curvature of Finsler metrics of scalar curvature, **Journal of the London Math. Society**, v. 68, p. 762-780. 2003.

[2] DOUGLAS, J. The general geometry of paths, **Ann. Math.**, v. 29, p. 143-168. 1928.

[3] HUANG, L.; MO, X. On spherically symmetric Finsler metrics of scalar curvature, **J. Geom. Phys.**, v. 62, p. 2279-2287. 2012.

[4] HUANG, L.; MO, X. Projectively flat Finsler metrics with orthogonal invariance, **Ann. Polon. Math.**, v. 107, p. 259-270. 2013.

[5] MO, X.; ZHOU, L. The curvatures of spherically symmetric Finsler metrics in $\mathbb{R}^n$, **arXiv:**1202.4543v4 [math.DG], Oct. 2014.

[6] MO, X.; SOLORZANO, N.; TENENBLAT, K. On spherically symmetric Finsler metrics with vanishing Douglas curvature, **Diff. Geom. Appl.**, v. 31, n. 6, p. 746-758. 2013.

[7] RUTZ, S. F. Symmetry in Finsler spaces, **Contemp. Math.**, v. 196, p. 289-300. 1996.

[8] SHEN, Z. **Differential Geometry of Spray and Finsler Spaces**, Dordrecht: Springer, 2001.

[9] SHEN, Z. **Lectures on Finsler Geometry**, Singapore: World Scientific Publishing Co., 2001.

[10] SHEN, Z. Projectively flat Finsler metrics of constant flag curvature, **Trans. AMS**, v. 355, p. 1713-1728. 2003.

[11] SHEN, Z. Volume Comparison and Its Applications in Riemann-Finsler Geometry, **Advances in Mathematics**, v. 128, n. 2, p. 306-328. 1997.

[12] YU, C.; ZHU, H. On a new class of Finsler metrics, **Diff. Geom. Appl.**, v. 29, p. 244-254. 2011.

[13] ZHOU, L. Spherically symmetric Finsler metrics in $\mathbb{R}^n$, **Publ. Math. Debrecen**, v. 80, n. 1-2, p. 67-77. 2012.

[14] ZHOU, L. The spherically symmetric Finsler metrics with isotropic S-curvature, **Journal of Math. Analysis and Appl.**, v. 431, n. 2, p. 1008-1021. 2015.

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