Laguerre Type Surfaces

Superfícies de tipo Laguerre

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Abstract: In this work we introduce the Laguerre type surfaces, this class of surfaces contains the minimal surfaces, another class of these surfaces are the Weingarten surfaces of Laguerre type. We show that all graph-type harmonic surfaces are Laguerre type surfaces. As an application we classify the graph-type harmonic surfaces that are Weingarten surfaces of Laguerre type and also we classify the graph-type harmonic surfaces that are helicoidal. We present families of cyclic graph-type harmonic surfaces with the planes of the foliation parallel. Finally, we give a classification of translation surfaces of Laguerre type.

Keywords: Laguerre minimal surface. Weingarten surface. Harmonic surface.

Resumo: Neste trabalho introduzimos as superfícies de tipo Laguerre, esta classe de superfícies contem as superfícies mínimas, uma outra classe de essas superfícies são as superfícies Weingarten de tipo Laguerre. Mostramos que toda superfície harmônica de tipo gráfico é uma superfície de tipo Laguerre. Como aplicação classificamos as superfícies harmônicas de tipo gráfico que são superfícies de Weingarten de tipo Laguerre e também classificamos as superfícies harmônicas de tipo gráfico que são helicoidais. Apresentamos famílias de superfícies harmônicas de tipo gráfico cíclicas com planos da foliação paralelos. Finalmente, fornecemos uma clasificação das superfícies translação de tipo Laguerre.

Palavras-chave: Superície mínima de Laguerre. Superfície de Weingarten. Superfície harmônica.

1   Introduction

A regular parametrized surface is harmonic if its coordinate functions are harmonic, a class of these surfaces are the minimal surfaces that has been extensively studied by

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many authors, a compilation of results on minimal surfaces can be found in [14]. In [8], the author studies geometric properties of the harmonic immersions and in [19] the authors present a special class of harmonic surfaces called graph-type harmonic surfaces and show that a graph-type harmonic surface is minimal if and only if it is part of a plane or a helicoid.

A surface $M \subset \mathbb{R}^3$ is a *Laguerre minimal surface* if

$$\triangle_{III} \left(\frac{H}{K} \right) = 0,$$

where H is the mean curvature, K is the Gaussian curvature and III is the third fundamental form of M. Laguerre minimal surfaces have been studied by several authors, for example see [5], [15], [21], [22].

Let $M \subset \mathbb{R}^3$ be a regular surface with Gauss map $N : M \to \mathbb{S}^2$, we say that M is a *graph-type surface* if $N(M)$ it is contained in an open spherical cap. Let $\sigma \in \mathbb{R}^3$ be a plane, we define the *projection application* $\pi_\sigma : \mathbb{R}^3 \to \sigma$ as the orthogonal projection of p into σ. Motivated by the work on Laguerre minimal surfaces, we introduced the class of *Laguerre type surface* as being a graph-type surface M in the Euclidean space with Gauss map N, mean curvature H and Gaussian curvature K, such that there exists a plane σ passing through the origin, such that

$$\triangle_L \left(\frac{h_\sigma H}{K} \right) = 0,$$

where L is the metric in σ induced by the application $\pi_\sigma|_M$, $N(M)$ is contained in an open spherical cap determined by σ and $h_\sigma = \pi_\sigma \circ N$.

This class of surfaces contains the minimal surfaces. Another special class of Laguerre type surfaces are the surfaces such that

$$\frac{h_\sigma H}{K} = c,$$

where c is a constant, which will be called *Weingarten surfaces of Laguerre type*.

A graph of a function of n variables f is called *translational hypersurface* if, f can be written as a sum of functions of type $f(u_1, \cdots, u_n) = f_1(u_1) + \cdots + f_n(u_n)$. In the last years several authors studied translational hypersurfaces and the hypersurfaces translational Weingarten type (see [1], [3], [10], [12], [20]). In [4] the authors classified the translational surfaces of Weingarten type in the three-dimensional space. Recently in [2], the authors studied the translation surfaces of

linear Weingarten type in three-dimensional space. In [9], the authors classified the translational hypersurfaces of Euclidean space with constant scalar curvature.

In this work we show that all graph-type harmonic surfaces is a Laguerre type surface. As an application we classify the graph-type harmonic surfaces that are Weingarten surfaces of Laguerre type and classify the graph-type harmonic surfaces that are helicoidal. A surface M is said to be cyclic if is determined by a smooth 1-parameter family of circles. By using the works [6], [13], [16], López in [11] showed that a cyclic linear Weingarten is a subset of around sphere or the planes of the foliation are parallel. We present families of cyclic graph-type harmonic surfaces with the planes of the foliation parallel. Finally, we give a classification of translation surfaces of Laguerre type that generate examples of Laguerre type surfaces that are not harmonic surfaces.

2   Preliminaries

In this section we recall some results and definitions which will be necessary in the following sections. Let Ω be an open subset of $I\!R^n$ and $u = (u_1, u_2, \cdots, u_n) \in \Omega$. Let $X : \Omega \subset I\!R^n \to I\!R^{n+1}$, $n \geq 3$, be a hypersurface parameterized by lines of curvature, with distinct principal curvatures λ_i, $1 \leq i \leq n$ and let $N : \Omega \subset I\!R^n \to I\!R^{n+1}$ be a unit normal vector field of X. Then

$$\langle X_{,i}, X_{,j} \rangle = \delta_{ij} g_{ii} , \ 1 \leq i, j \leq n , \tag{1}$$

$$N_{,i} = -\lambda_i X_{,i} , \tag{2}$$

where the subscript ",i" denotes the derivative with respect to u_i. Moreover,

$$X_{,ij} - \Gamma_{ij}^i X_{,i} - \Gamma_{ij}^j X_{,j} = 0 , \ \ 1 \leq i \neq j \leq n , \tag{3}$$

where Γ_{ij}^k are the Christoffel symbols.
The Christoffel symbols in terms of the metric (1) are given by

$$\Gamma_{ij}^k = 0, \quad \Gamma_{ii}^i = \frac{g_{ii,i}}{2g_{ii}}, \quad \Gamma_{ii}^j = -\frac{g_{ii,j}}{2g_{jj}}, \quad \Gamma_{ij}^i = \frac{g_{ii,j}}{2g_{ii}}, \tag{4}$$

where i, j, k are distinct.

We now consider the higher-dimensional Laplace invariants of the system of

equations (3) (see [7] for the definition of these invariants)

$$\begin{aligned} m_{ij} &= -\Gamma^i_{ij,i} + \Gamma^i_{ij}\Gamma^j_{ij}, \\ m_{ijk} &= \Gamma^i_{ij} - \Gamma^k_{kj} \ , \ \ k \neq i,j \ , \ \ 1 \leq k \leq n. \end{aligned} \tag{5}$$

In this paper the inner product $\langle,\rangle : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ is defined by

$$\langle f, g \rangle = f_1 g_1 + f_2 g_2, \ \text{where} \ f = f_1 + if_2, \ g = g_1 + ig_2,$$

are holomorphic functions. In the computation we use the following properties: If $f, g, h : \mathbb{C} \to \mathbb{C}, \ z = u_1 + iu_2 \in \mathbb{C}$ are holomorphic functions then

$$\langle f, g \rangle_{,u_1} = \langle f', g \rangle + \langle f, g' \rangle, \ \ \langle f, g \rangle_{,u_2} = \langle if', g \rangle + \langle f, ig' \rangle, \ \ \langle fg, h \rangle = \langle g, \overline{f}h \rangle,$$

$$\triangle \langle f, g \rangle = 4\langle f', g' \rangle, \ \ \langle f, g \rangle + i\langle f, ig \rangle = f\overline{g}, \ \ \langle 1, f \rangle \langle 1, if \rangle = \frac{1}{2}\langle 1, if^2 \rangle, \tag{6}$$

$$\langle 1, f \rangle^2 - \langle 1, if \rangle^2 = \langle 1, f^2 \rangle.$$

The following theorem was obtained in [17].

Theorem 1. *Let $M^n \subset \mathbb{R}^{n+1}, n \geq 3$, be a hypersurface parametrized by lines of curvature, with n distinct principal curvatures $-\lambda_i$.*
1) The foliations of M^n are umbilical hypersurfaces in M^n if and only if $m_{ijk} = 0, \ \forall \, 1 \leq i \neq j \neq k \leq n$,
2) The foliations of M^n are Dupin hypersurfaces in M^n if and only if $m_{ij} = 0, \ \forall \, 1 \leq i \neq j \leq n$,
3) If $m_{ij} = 0$. Then the lines of curvature have constant geodesic curvature.

Remark 1. *The item (3) of Theorem 1, is true for $n \geq 2$.*

Definition 1. *A surface $M \subset \mathbb{R}^3$ is called* Translation surface *if locally it can be parametrized by*

$$X(u_1, u_2) = (u_1, u_2, f_1(u_1) + f_2(u_2)). \tag{7}$$

The Gauss map is given by

$$N = \frac{1}{\Delta}\left(-f_1', -f_2', 1\right), \tag{8}$$

where

$$\Delta = \sqrt{1 + (f_1')^2 + (f_2')^2}. \tag{9}$$

The mean curvature H and the Gaussian curvature K are given by

$$H = \frac{(1 + (f_2')^2)f_1'' + (1 + (f_1')^2)f_2''}{2\Delta^3}, \quad K = \frac{f_1'' f_2''}{\Delta^4}. \tag{10}$$

The following result was obtained in [18].

Lemma 1. *Let $X = (g, 0)$ be an orthogonal parametrization of $\Pi = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : u_3 = 0\}$ where $g : \mathbb{C} \to \mathbb{C}$ be a holomorphic function and m_{12} is the Laplace invariant given in (5). Then*
$m_{12} = 0$ if and only if g is given by

$$g(z) = \frac{z_1 z + z_2}{z_3 z + z_4} \quad or \quad g(z) = \frac{z_1 e^{\sqrt{-2c}z} + z_2}{z_3 e^{\sqrt{-2c}z} + z_4}, \ z_1 z_4 - z_2 z_3 \neq 0. \tag{11}$$

where $z_i \in \mathbb{C}$, $c \in \mathbb{R}$. Moreover, in this case $m_{12} = 0$ if and only if $m_{21} = 0$.

The following proposition is a consequence of Theorem 1 (item 3)) and the Lemma 1.

Proposition 1. *Let $X = (g, 0)$ be an orthogonal parametrization of $\Pi = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : u_3 = 0\}$ where $g : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Then*
The coordinate curves are circles or straight lines if and only if $g(z)$ is given by (11).

Definition 2. *Let $X(u_1, u_2) = (\varphi_1(u_1, u_2), \varphi_2(u_1, u_2), \varphi_3(u_1, u_2))$ be a parametrization of the surface M in $\mathbb{R}^3$, M is called* harmonic surface *if $\triangle \varphi_i = 0$, for $i = 1, 2, 3$.*

Definition 3. *Let $f, g : \mathbb{C} \to \mathbb{C}$ be holomorphic functions. The surface M in $\mathbb{R}^3$ parametrized by $X(z) = (g(z), < 1, f(z) >)$ is called* graph-type harmonic surface.

The next proposition was obtained in [19].

Proposition 2. *Let $M \subset \mathbb{R}^3$ be a graph-type harmonic surface given by*

$$X(z) = (g(z), < 1, f(z) >). \tag{12}$$

The Gauss map is given by

$$N = \frac{1}{D}(-g'\bar{f}', |g'|^2), \tag{13}$$

where

$$D = |g'|\sqrt{|g'|^2 + |f'|^2}.$$

The coefficients of the first fundamental form are given by

$$E = |g'|^2 + <1, f'>^2, \quad F = -\frac{1}{2}<i, (f')^2>, \quad G = |g'|^2 + <1, if'>^2, \quad (14)$$

and the regularity condition is given by $g'(z) \neq 0$.
The mean curvature H and the Gaussian curvature K are given by

$$H = -\frac{|g'|^2 <A, (f')^2>}{2D^3} \qquad K = -\frac{|g'|^4|A|^2}{D^4}, \qquad (15)$$

where

$$A = f'' - f'\frac{g''}{g'}. \qquad (16)$$

3 Laguerre type surfaces

In this section we consider $\sigma = \{(u_1, u_2, u_3) \in \mathbb{R}^3 / u_3 = 0\}$ and the open spherical cap determined by σ as $S^\sigma = \{u = (u_1, u_2, u_3) \in \mathbb{R}^3 / |u| = 1, u_3 > 0\}$.

Consider a plane $\sigma \subset \mathbb{R}^3$, we define the *projection application* $\pi_\sigma : \mathbb{R}^3 \to \sigma$ as being the orthogonal projection of p into σ.

Definition 4. *Let M be a surface in $\mathbb{R}^3$ with Gauss map N, mean curvature H, Gaussian curvature K. The surface M is called* Laguerre type surface *if there is a plane σ passing through the origin such that*

$$\triangle_L \left(\frac{h_\sigma H}{K} \right) = 0,$$

where L is the metric in σ induced by the application $\pi_\sigma|_M$, $N(M)$ is contained in S^σ and $h_\sigma = \pi_\sigma \circ N$.

This class of surfaces includes the minimal surfaces. Another special class of the Laguerre type surfaces are the surfaces such that

$$\frac{h_\sigma H}{K} = c,$$

where c is a constant. These surfaces will be called *Weingarten surfaces of Laguerre type*.

Theorem 2. *All graph-type harmonic surfaces whit Gaussian curvature non zero is a Laguerre type surface.*

Proof. We consider the plane $\sigma = \{(u_1, u_2, u_3) \in \mathbb{R}^3/u_3 = 0\}$. From (6), (13) and (15) we obtain

$$\frac{h_\sigma H}{K} = \frac{1}{2} \left\langle 1, \frac{(f')^2}{A} \right\rangle, \tag{17}$$

hence, it follows the result. □

We remark that by Theorem 2, the graph-type harmonic surfaces are a special class of Laguerre type surfaces. In the following we will present some results of classification of these surfaces with additional conditions.

The following result classifies the graph-type harmonic surfaces that are Weingarten surfaces of Laguerre type.

Proposition 3. *A graph-type harmonic surface given by (12) is a Weingarten surface of Laguerre type, with $c \neq 0$ if and only if, up to translation, X is a surface of rotation, given by*

$$X(z) = (2ce^{u_1}\cos u_2, 2ce^{u_1}\sin u_2, -2cu_1). \tag{18}$$

Proof. Using (17), we get that X is a Weingarten surface of Laguerre type, if and only if, $(f')^2 = 2cA$, now from (16), it follows

$$f' = 2c\left(\frac{f''}{f'} - \frac{g''}{g'}\right),$$

integrating this equation we obtain

$$g = 2cz_1 e^{\frac{-f}{2c}} + z_2.$$

If $f = u(u_1, u_2) + iv(u_1, u_2)$, then X is given by

$$X(z) = \left(2ce^{a - \frac{u(u_1,u_2)}{2c}}e^{i(\frac{b - v(u_1,u_2)}{2c})} + z_2, u(u_1, u_2)\right),$$

where $z_1 = e^{a+ib}$, thus, up to translation and a change of variable we obtain the expression (18). In Figure 1, we have an example of Weingarten surface of Laguerre type. □

Proposition 4. *A graph-type harmonic surface given by (12) is a helicoidal surface, if and only if, X is given by*

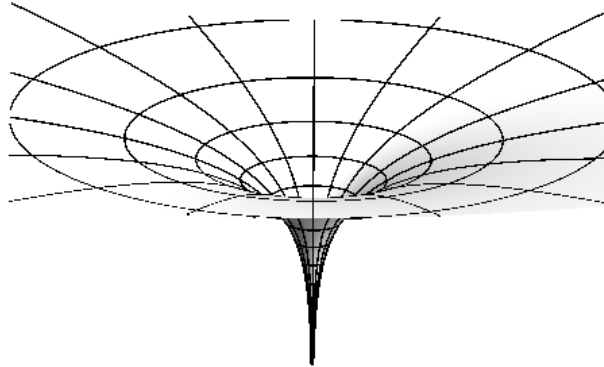$$X(z) = (e^{u_1}\cos u_2, e^{u_1}\sin u_2, au_1 + bu_2 + c).$$

Figure 1: $c = 1$

Proof. A graph-type harmonic surface is helicoidal if

$$X(z) = (e^{u_1}\cos u_2, e^{u_1}\sin u_2, \lambda(u_1) + bu_2),$$

where $\lambda(u_1)$ is a real function, since, $\lambda(u_1) + bu_2$ must be a harmonic function, we have that $\lambda''(u_1) = 0$, by integration it follows the result. In the Figure 2, we have an example of helicoidal graph-type harmonic surface. $\qquad\square$
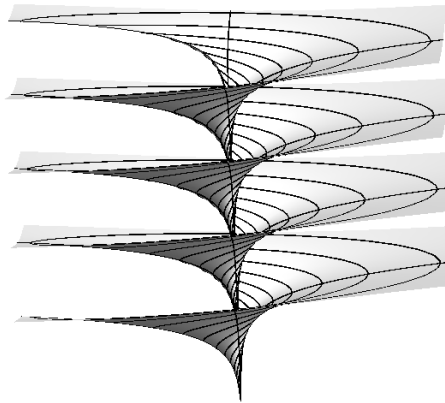

Figure 2: $a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{1}{5}$

Proposition 5. *Let M be a graph-type harmonic surface given by (12). If $X_1(z) = (g(z), c_1u_1)$ or $X_2(z) = (g(z), -c_1u_2)$ and $g(z)$ is given by (11). Then the surfaces are ruled with lines parallel to the u_1u_2 plane or cyclic with circles parallel to the u_1u_2 plane.*

Proof. From Proposition 1, we have that the coordinate curve $X_1(u_1^0, u_2) = (g(u_1^0, u_2), c_1u_1^0)$ where $u_1^0 =$constant are straight lines or circles. Similarly, the coordinate

curve $X_2(u_1, u_2^0) = (g(u_1, u_2^0), c_1 u_2^0)$ where $u_2^0 =$constant are straight lines or circles, hence it follows the result. In the Figures 3 and 4, we have examples of cyclic graph-type harmonic surfaces. $\square$
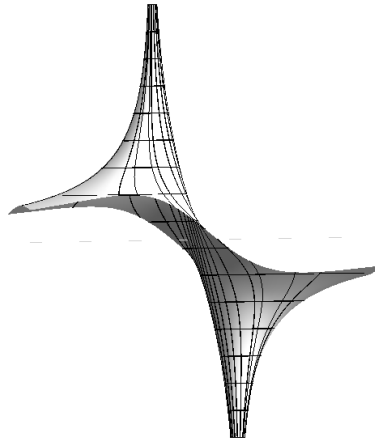
Figure 3: $f(z) = -z$, $g(z) = \frac{(1+i)e^{\sqrt{-2}z}+1}{(1/2+i)e^{\sqrt{-2}z}+1+i}$
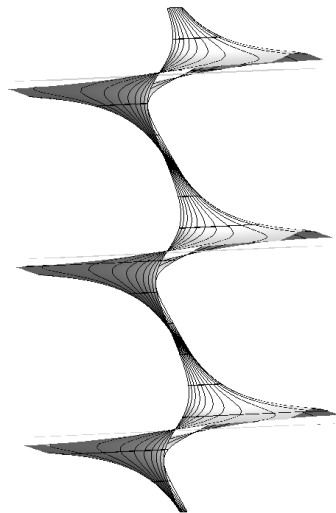
Figure 4: $f(z) = z$, $g(z) = \frac{(1+i)e^{\sqrt{-4}z}}{(1+i)e^{\sqrt{-4}z}+1}$

Proposition 6. *Let X be a translation surface given by (7). Then X is a Laguerre type surface, if and only if, $f_1(u_1)$ and $f_2(u_2)$ are given by*
i)

$$f_1(u_1) = au_1^2 + a_2u_1 + a_3 \text{ or } f_1(u_1) = c_2 + \int \tan\left[\int \frac{1}{au_1^2 + a_2u_1 + a_3} du_1 + c_1\right] du_1,$$

$$f_2(u_2) = au_2^2 + b_2u_2 + b_3 \text{ or } f_2(u_2) = c_4 + \int \tan\left[\int \frac{1}{-au_2^2 + b_2u_2 + b_3} du_2 + c_3\right] du_2,$$

where $a \neq 0$.

ii)

$$f_1(u_1) = c_2 - 2a_3 \ln\left[\cos\left(\frac{u_1}{2a_3} + c_1\right)\right] \text{ or } f_1(u_1) = c_4 + \int \tan\left[\frac{\ln(a_2u_1 + a_3)}{a_2} + c_3\right] du_1,$$

$$f_2(u_2) = c_6 - 2b_3 \ln\left[\cos\left(\frac{u_2}{2b_3} + c_5\right)\right] \text{ or } f_2(u_2) = c_8 + \int \tan\left[\frac{\ln(b_2u_1 + b_3)}{b_2} + c_7\right] du_2,$$

where a_2, a_3, b_2, b_3 are non-zero real constants.

Proof. From (8)-(10), X given by (7) is a Laguerre type surface, if and only if

$$\triangle \left(\frac{1 + (f_1')^2}{2f_1''} + \frac{1 + (f_2')^2}{2f_2''}\right) = 0,$$

this equation is equivalent to

$$\left(\frac{1 + (f_1')^2}{f_1''}\right)'' = 2a, \quad \left(\frac{1 + (f_2')^2}{f_2''}\right)'' = -2a. \tag{19}$$

If $a \neq 0$, trivial solutions of these equations are given by

$$f_1(u_1) = au_1^2 + a_2u_1 + a_3, \quad f_2(u_2) = -au_2^2 + b_2u_2 + b_3.$$

Other solutions are obtained by integrating the following equations

$$\frac{f_1''}{1 + (f_1')^2} = \frac{1}{au_1^2 + a_2u_1 + a_3}, \quad \frac{f_2''}{1 + (f_2')^2} = \frac{1}{-au_2^2 + b_2u_2 + b_3}.$$

We remark that $\dfrac{f_1''}{1 + (f_1')^2} = (\arctan f_1')'$, using this fact we obtain the item i).

If $a = 0$, the solutions of (19) are obtained integrating the equations

$$\frac{f_1''}{1 + (f_1')^2} = \frac{1}{a_2u_1 + a_3}, \quad \frac{f_2''}{1 + (f_2')^2} = \frac{1}{b_2u_2 + b_3}.$$

Thus, considering the cases where $a_2 = 0$ $(a_2 \neq 0)$ and $b_2 = 0$ $(b_2 \neq 0)$ we obtain item ii). $\qquad \square$

Remark 2. *We remark that the surfaces of item ii) are Weingarten surfaces of Laguerre type and include the Scherk minimal surface when $a_3 = -b_3$. In the Figures 5, 6, 7 and 8 (Scherk minimal surface), we have examples of translation Laguerre type surfaces, these examples are not harmonic surfaces.*
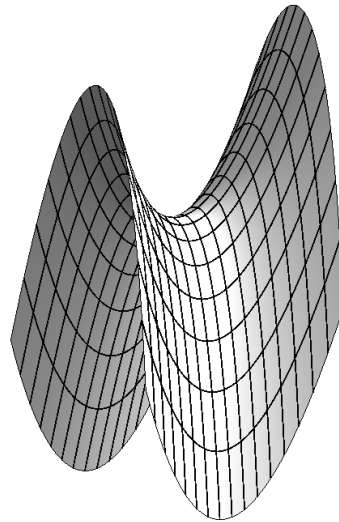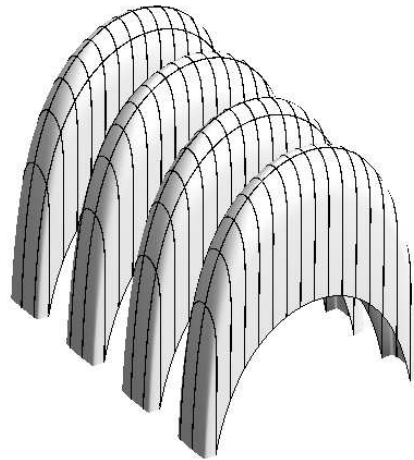

Figure 5: $f_1(u_1) = \frac{1}{2}u_1^2 + \frac{1}{3}u_1 + \frac{1}{6}$, $f_2(u_2) = -\frac{1}{2}u_2^2 + \frac{1}{2}u_2 - \frac{1}{4}$
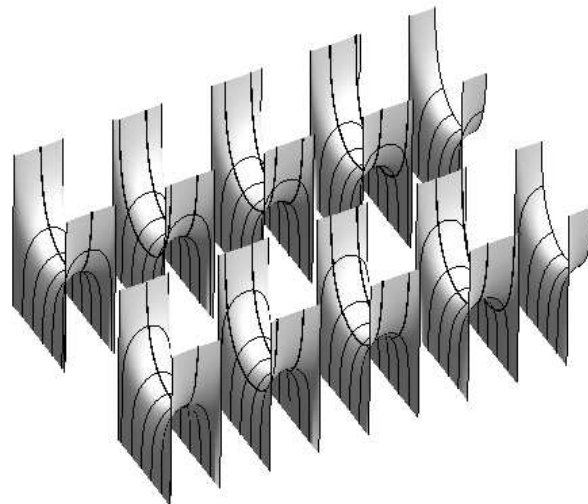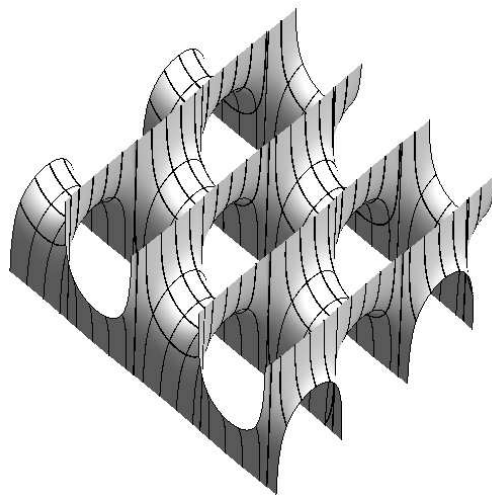

Figure 6: $f_1(u_1) = \ln[\cos(-u_1 + 1)]$, $f_2(u_2) = -2\ln[\cos(\frac{u_2}{2} + 2)]$

Remark 3. *In this work we present an initial theory about the Laguerre type surfaces, the study of these surfaces with additional geometric or analytical properties will be object of study and we hope to present future works in this sense.*

Figure 7: $f_1(u_1) = \pi + 2e\ln[\cos(\frac{u_1}{2e} + 1)]$, $f_2(u_2) = \ln[\cos(-u_2 + 2)]$

Figure 8: $f_1(u_1) = 1 + \ln[\cos(-u_1 + 1)]$, $f_2(u_2) = 1 - \ln[\cos(u_2 + 2)]$

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