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The Dandelin Spheres and the Method of the Conic Sections of the Greeks

As esferas de Dandelin e o método das seções cônicas dos gregos

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Abstract: Many Greek mathematicians have devoted themselves to the study of the conic sections, however, it is due to Apollonius of Perga the most known systematization on this subject. The Apollonius' approach can be found in a treatise entitled *Cônics*. However, the definitions of the conic sections currently used were proved in the nineteenth century by Dandelin and Quetélet. They introduced a new idea related to the demonstration of the properties of the conics. They have proved that, given a plane that cuts a cone, there are one or two spheres (Dandelin spheres) that are simultaneously tangent to the plane and the cone, depending on the angle of the section. It is common to present the parabolas as a locus of points that depend on the relation between the distance of these points to the focus and the directrix. This relationship is known as the *focus-directrix* property. However, it is not common to establish, geometrically, this type of relationship in the study of ellipses and hyperboles. In the present work through strictly geometric procedures and the Dandelin-Quetélet theorem, we show in a different perspective the equivalence between the focus-directrix property and the method of the conic sections of the Greeks.

Keywords: Conic sections. Apollonius. Dandelin. Quetélet.

Resumo: Muitos matemáticos gregos dedicaram-se ao estudo das seções cônicas, porém, deve-se a Apolônio de Perga a sistematização mais conhecida sobre este tema. A abordagem de Apolônio pode ser encontrada em um tratado intitulado *Cônicas*. No entanto, as definições das seções cônicas atualmente utilizadas, foram provadas no século XIX, por Dandelin e Quetélet. Eles introduziram uma nova ideia relacionada à demonstração das propriedades das cônicas, mostrando que dado um plano que secciona um cone, existem uma ou duas esferas (esferas de Dandelin) que são simultaneamente tangentes ao plano e ao cone, dependendo do ângulo da seção. É comum apresentar as parábolas como um

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lugar geométrico de pontos que dependem da relação entre a distância destes ao foco e à reta diretriz. Esta relação é conhecida como propriedade *foco - diretriz*. No entanto, não é comum se estabelecer, geometricamente, este tipo de relação no estudo das elipses e das hipérboles. No presente trabalho, por meio de procedimentos estritamente geométricos e do teorema de Dandelin-Quetélet, mostra-se, em uma diferente perspectiva, a equivalência entre a propriedade foco - diretriz e o método das seções cônicas dos gregos.

Palavras-chave: Seções cônicas. Apolônio. Dandelin. Quetélet.

1 Introduction

One does not know for sure when man's interest in geometric forms began, but the importance of this interest for the development of mathematics, and consequently for the development of mankind, is an incontestable fact. One of the earliest Greek problems known as Delian problem (or doubling the cube) consists in construct a cube whose volume was twice the volume of a given cube, using only the non-graduated ruler and the compass. Now we know that this problem has no solution (as well as the problems of squaring the circle and trisecting the angle). This was a problem that aroused the attention of many ancient Greeks, and from the attempts to find a solution came the curves which we now know as parabolas, ellipses, and hyperbolas. Some writers ascribe to Menaechmus the discovery of these curves, which he called the *section of a cone of acute angle* (ellipse), *section of a cone of right angle* (parabola), and *section of a cone of obtuse angle* (hyperbola).

Although Menaechmus has the credit for the discovery, many other ancient Greek mathematicians were interested in the study of these curves, including Euclid and Apollonius of Perga, which is one of the most important names appearing in this context. In Book XI of the Elements, Euclid defined a cone as *a solid generated by a rectangle triangle rotating around one of the cathetus*. Thus, ellipses, parabolas and hyperbolas were the curves obtained from the cut of a straight circular cone by a plane perpendicular to an element of the cone (the elements of the cone are understood as the sides of the triangle that generate the cone).

According to Boyer [5], Apollonius was born in the city of Perga (South Asia Minor) and would have lived in the period of 262 BC to 190 BC. He probably grew up in Alexandria where he taught at a university. He was a mathematician and astronomer, but most of his work was lost. Only two works survives: *Cutting-off of a Ratio* and the treatise on *Conics* (in Greek $K\omega\nu\iota\kappa\acute{\alpha}$). The first one was known only in Arabic until 1706, when Edmund Halley published a Latin translation. The latter one was very famous in ancient Greece and rendered Apollonius the title of *Great Geometer*. The Conics was composed of eight books, but only the first four of them survived originally in Greek. Books V, VI and VII survived because they were translated into Arabic by *Thabit ibn Qurra* (1826 - 1901), but the book VIII was lost. Fortunately, according to Michael N. Fried [4] Edmond Halley's provides a reconstruction of Book VIII. Fried provides the first complete English translation of Halley's reconstruction of this Book (including supplementary notes). He emphasizes:

> The work comprised eight books, of which four have come down to us in their original Greek and three in Arabic. By the time the Arabic translations were produced, the eighth book had already been lost. In 1710, Edmond Halley, then Savilian Professor of Geometry at Oxford, produced an edition of the Greek text of the Conics of Books I-IV, a translation into Latin from the Arabic versions of Books V-VII, and a reconstruction of Book VIII. [4]

According to Carl Boyer [5], the conic sections were already known more than a century before Apollonius to write his treatise, however Apollonius' Conics is considered one of the greatest works of advanced mathematics in antiquity.

Some of Apollonius works came to us through other authors who somehow, preserved parts or the whole contents. Much of what remains of the Apollonius' works came even through the works of Pappus of Alexandria (circa 320). The most important work of Pappus is a mathematical collection titled *Synagoge* (translated as Collection) which is very rich in historical records of works of a several number of Greek mathematicians as Euclid, Archimedes, Apollonius, and Ptolemy. Some lost works of these (and others) authors were reproduced by him, but, his work also includes new discoveries and generalizations not found in any earlier work. Unfortunately, the first book and the first part of the second book of the *Collection* are now lost.

Six of Apollonius' works were included along with two of the most advanced treatises of Euclid's work, in a Pappus collection titled *Treasury of Analysis*, now lost too. It is believed that *Treasury* should have included the essence of what we now call *Analytic Geometry*, and, according to Boyer "*Pappus described this as a special body of doctrine for those who, after going through the usual elements, wish to be capable of solving problems involving curves*" ([5], p. 128).

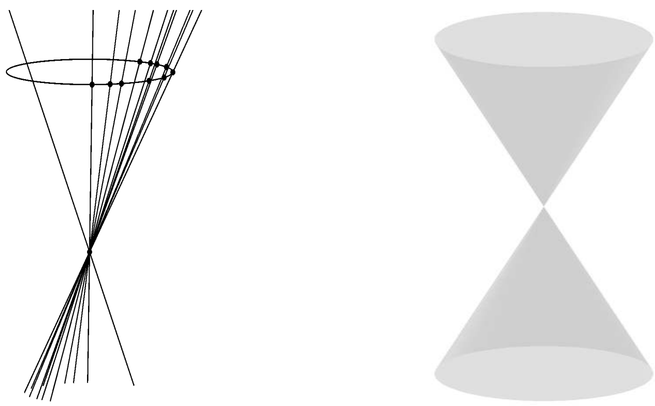
One of the main advancements provided by Apollonius was the demonstration that three types of cones would not be required to determine the conic sections. He showed that from a single cone, we can get all three conic sections by varying the slope of the secant plane.

> Before the time of Apollonius, ellipses, the parabolas, and the hyperbolas were derived as sections of three distinctly different types of right circular cones, according as the vertex angle was acute, right, or obtuse. Apollonius, apparently for the first time, systematically showed that it

> is not necessary to take sections perpendicular to an element of the cone and that from a single cone one can obtain all three varieties of conic sections simply by varying the inclination of the cutting plane. This was an important step in linking the three types of curve. A second important generalization was made when Apollonius demonstrated that the cone need not be a right cone - that is, one whose axis is perpendicular to the circular base - but can equally well be an oblique or scalene circular cone. [5]

We also owe to Apollonius the replacement of the single cone by a circular double-napped cone. Boyer quotes Apollonius to show that his definition of circular cone is the same as that used today:

> If a straight line indefinite in length and passing always through a fixed point be made to move around the circumference of a circle which is not in the same plane with the point so as to pass successively through every point of that circumference, the moving straight line will trace out the surface of a double cone. [5]



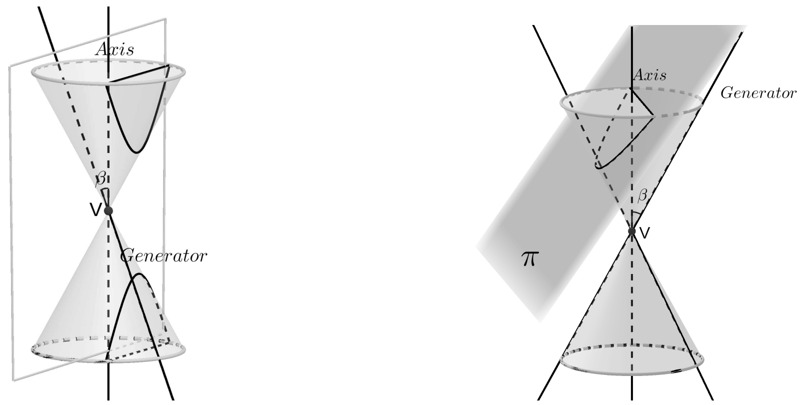
(a) (b)

Figure 1: The double-napped cone generated by a moving straight line around a circumference of a circle.

2 The conic sections

In this section we will present a resume of the conic sections as the curves obtained from the intersection between a plane and a double-napped cone generated by a line as described by Apollonius (see Figure 1). The circumference of the Apollonius' definition is called *directrix*, the fixed point is called *vertex*, the lines that move along the circumference are called *generators*, and the straight line joining the vertex to the centre of the circumference (directrix) is called the *axis* of the cone.

For simplicity we will consider a double right circular cone generated by the rotation of the line h (generator) about the axis e in an angle β such as the vertex V is the point of intersection between e and h. We will also consider a plane π (secant plane) intersecting the cone in an angle α, as shown in Figures 2 and 3, and show that these intersections can be a *hyperbola*, a *parabola* or an *ellipse*.



(a) Hyperbola **(b)** Parabola

Figure 2: If $\alpha = 0$ and $\beta > \alpha$ the conic section is a hyperbola; If $\alpha = \beta$ the conic section is a parabola.

If the π plane is parallel to the axis we have $\alpha = 0$ and $\beta > \alpha$. In this case the conic section is a hyperbola (Figure 2a); If the π plane is parallel to a generator we have $\alpha = \beta$, and in this case the conic section is a parabola (Figure 2b); If the π plane intersects all the generators we have $\alpha > \beta$ and the conic section is an ellipse (Figure 3).

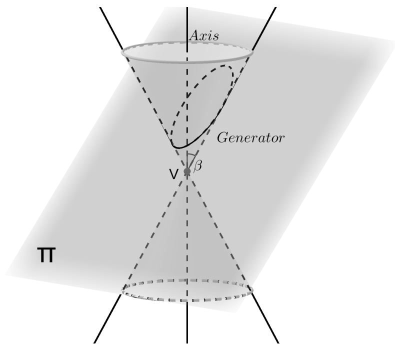


Figure 3: If the π plane intersects all the generators we have $\alpha > \beta$ and the conic section is an ellipse.

It is important to note that the intersections of a right circular cone with a plane passing through the vertex can generate: *a point* (degenerate ellipse) if the plane passes through the vertex of the cone and does not contain any generatrix, *a line* (degenerate parabola) if the plane contains a single generatrix, *two intersecting lines* (degenerate hyperbola) if the plane contains *two generatrix* of the cone. A circumference is a particular case of the ellipse, obtained by the intersection of the right circular cone with a plane perpendicular to the axis. In the present text we will deal only with the non - degenerate conic sections.

2.1 The focus-directrix property

The relationship between the focus and the directrix of a conic section is known as the *focus-directrix properties* (the term focus was first used by Johannes Kepler in 1604). According to Boyer, although Apollonius seems to know the focus-directrix properties of the conic sections, in his treated *Conics*, he makes no mention of these properties applied to parabolas. Boyer states that, probably, the earliest mention of these relations is due to Pappus, in his book VII of the *Collection* (Synagoge).

> Book VII of the Collection contains the first statement on record of the focus-directrix property of the three conic sections. It appears that Apollonius knew of the focal properties for central conics, but it is possible that the focus-directrix property for the parabola was not known before Pappus. [5]

Papus also observed that if the ratio of distances to a fixed point and a fixed line is a constant less than 1 the locus of the points was an ellipse, and if this ratio was higher than 1 the conic section should be a hyperbola. These properties were probably known in the time of Diocles and Apollonius. Some authors has stated that Diocles (240 BC - 180 BC) showed how to construct a parabola using the focus-directrix property.

It is usual today to work with the conic sections as a locus of points satisfying the focus-directrix properties using algebraic methods rather geometric methods.

We will use the Charles Taylor book *An Introduction to the ancient and modern geometry of conics* (1881) [8] to give a summarized idea of the conic sections trough a geometric method.

From now on we will refer to a *conic section* simply by *conic* and we will show that the conic sections coincides with the locus of the points of a plane whose ratio of distances to a fixed point and a fixed straight line is constant. The fixed point is called *focus*; the fixed straight line *directrix*, and the constant ratio *eccentricity*. Thus, a conic is an ellipse, a parabola, or a hyperbola if the eccentricity is less than, equal to, or greater than unity, respectively. The axis of a conic is the straight line passing through the focus orthogonal to the directrix, and the point of intersection between the conic and the axis is called the vertex.

When the eccentricity is either greater or less than unity, the conic cuts its axis in a second point, which is called a vertex too. In such cases the term axis may denote the segment of line with extremals in the both vertices, and its middle point is called the *centre* of the conic. A conic with a centre is called a *central conic*.

A *chord* of a curve is a line segment joining any two distinct points on the curve. A *focal chord* of a conic is a chord passing through the focus. The chord of a conic that passes through the focus and is perpendicular to the axis is named the *latus rectum* (or parameter).

The locus of the middle points of a system of parallel chords is called a *diameter*. Each diameter have two *ends* or *extremities*, that are the points in which diameters and chords meet the curve. If a diameter of a central conic is orthogonal to the axis at the centre, then it is called a *minor* or *conjugate axis*.

When a diameter bisects the chords parallel to another diameter, both are called *conjugated diameters*, and two chords parallels to conjugate diameters is said to be *conjugate chords*.

A tangent to a conic can be understood as the limiting position of the secant line trough two points of the conic when its become infinitely close.

To understand the focus-directrix property we will present a construction of a conic whose focus, directrix and eccentricity are given.

Let F be the focus, and d the directrix of a conic. Let s be the line perpendicular to d at the point F and Let E be the foot of the perpendicular from the point F to the directrix d. In s take a point V between F and E. Since F, V and E are fixed points it is easy to see that $\frac{|FV|}{|VE|}$ is equal to a constant. This constant is the *eccentricity* e and the point V is the *vertex* of the conic. See Figure 4.

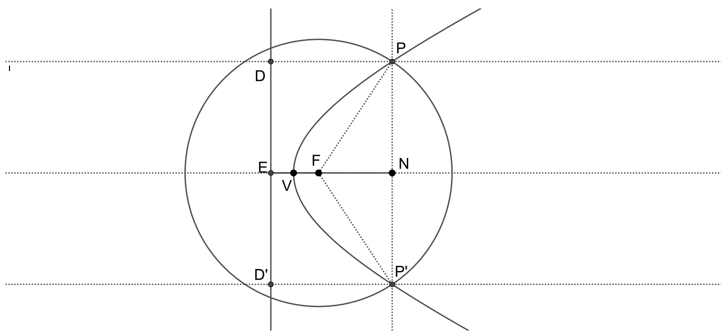


Figure 4: Construction of a general conic whose focus, directrix and eccentricity are given. The conic is determined by the position of the point V along the line s.

Now, take an arbitrary point N on the line s and consider a line t perpendicular to s at the point N. Draw a circle of center F and radius r satisfying the ratio $\frac{r}{|NE|} = \frac{|FV|}{|VE|}$. Thus, the points P and P' obtained from the intersections of the circle with the line t are such that

$$r = |FP| = |FP'| = |NE|\,\frac{|FV|}{|VE|} = e\,|NE|. \tag{1}$$

Let D and D' be the feet of the perpendiculars from the points P and P' to the directrix d, respectivelly. Note that $|PD| = |P'D'| = |NE|$ then, it follows from (1) that

$$\frac{|FP|}{|PD|} = \frac{|FV|}{|VE|} \Rightarrow \frac{|FP|}{|PD|} = e, \text{ and, } \frac{|FP'|}{|P'D'|} = e. \tag{2}$$

The line s is the *axis* of the conic. Thus, when the point N travels along the axis s the point P describe a conic of focus F, directrix d and eccentricity e. As P and P' are symetric with respect to the axis we can state that the conic is also simetric with respect the axis. Thus, if $P = P'$ then $N = V = P = P'$ and the secant line PP' becomes the tangent line at the vertex V.

It is easy to see that to perform this construction it is necessary and sufficient to guarantee that

$$|FN| \leq |FP|, \tag{3}$$

otherwise, the line t will not intersect the circle and the point P will not exist.

It follows from (3) and (1) that

$$|FN| \leq |FP| = |PD|\frac{|FV|}{|VE|} = |NE|\frac{|FV|}{|VE|},$$

since $|PD| = |NE|$. Then,

$$\frac{|FN|}{|NE|} \leq \frac{|FV|}{|VE|} = e. \tag{4}$$

Condition (4) together condition (3) enable us to determine if the conic is a parabola, an ellipse or a hyperbola, depending on the eccentricity $e = \frac{|FV|}{|VE|}$. Indeed, If $e \leq 1$, we must have $\frac{|FN|}{|NE|} \leq 1$, that is, $|FN| \leq |NE|$ and $|FN| \leq |FP|$. This situation determines two cases:

1. If $P \neq N$, $\forall N \neq V$, then $PN \neq 0$, $\forall N \neq V$. This tells us that the curve moves away from the axis when N moves away from V respecting (3), so the curve is a parabola. Note that this only occurs when $\frac{|FV|}{|VE|} = 1$.

2. If there exists V' such as $P = N = V'$ then N must be chosen between V and

V', and in this case the curve is an ellipse. This occurs when the ratio $\frac{|FV|}{|VE|}$ is less than 1.

when the ratio $e = \frac{|FV|}{|VE|}$ is greater than 1 we must have $\frac{|FN|}{|NE|} \leq e$, such as $e > 1$. This includes both cases $|FN| \leq |NE|$ and $|FN| > |NE|$. Condition (3) guarantees that this is only possible if there exists a second vertex V' on the axis such that N does not belong to the segment VV'. In this case the curve will be a hyperbola.

It follows from (2) that $\frac{|FV|}{|VE|} = 1 \Leftrightarrow |FP| = |PD|$. This result allows to define a parabola as the locus of points P of the plane moving in such a way that P is always equidistant from a fixed straight line (the directrix) and a fixed point (the focus).

Due to the fact that ellipses and hyperbolas have two vertices V and V' they both have two foci F_1 and F_2 and two directrices. In this case one say that the conic section is a *central conic*.

The focus-directrix property allows us to define a parabola as a locus of the points of a plane that satisfy a specific property.

In fact, it follows from (2) that $\frac{|FV|}{|VE|} = 1 \Leftrightarrow |FP| = |PD|$. This result allows to define the parabolas as the loci of points P of the plane moving in such a way that P is always equidistant from a fixed straight line (the directrix) and a fixed point (the focus).

This means that by taking a circle centered on a point P of a parabola and passing through the focus F, it will intercept the straight line d at a point D, such that the distance of D to the point P coincides with the distance of the focus F to the point P. This is equivalent to saying that points F, P and D determine a isosceles triangle FPD of base FD, where FD is the segment joining the points F and D, as shown in Figure 5.

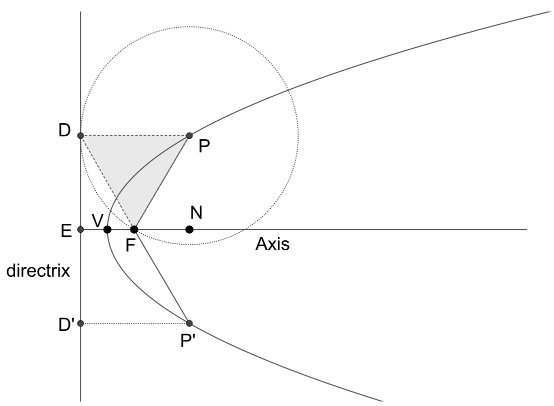


Figure 5: Parabola obtained from the construction of a general conic by putting the point V in a position such as $|FV|{=}|VE|$.

Note that if the conic section is a central conic it follows from (2) that $|PF_1| = e|PD|$ and $|F_2P| = e|PD_1|$, where D and D_1 are the feet of the perpendicular from P to the two directrices. In this case we have $|PF_1| + |PF_2| = e(|PD| + |PD_1|)$.

It is easy to see that $|PD| + |PD_1| = |DD_1|$, so $|DD_1|$ is a constant real number. Based on this and on (2) we can conclude that, if $0 \leq e < 1$, then $\frac{|FV|}{|VE|} < 1$, and therefore the conic section is an ellipse (see Figure 6).

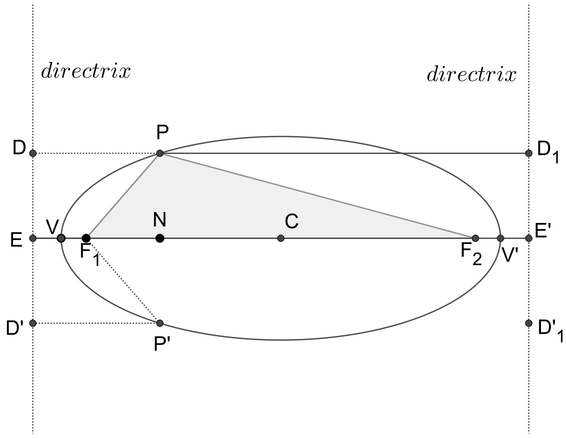


Figure 6: Ellipse obtained from the construction of a general conic by putting the point V in a position such as $|FV|<|VE|$.

Thus an ellipse can be defined as the locus of the points of a plane such that the sum of the distances to two fixed points (the foci) is constant.

In a similar way $|PF_1|-|PF_2| = e(|PD|-|PD_1|) = |DD_1|$ if $|PF_1| \geq |PF_2|$ and, $|PF_1| - |PF_2| = -e(|PD| - |PD_1|) = |DD_1|$ if $|PF_1| < |PF_2|$, i.e., $||PF_1| - |PF_2|| = e||PD| - |PD_1||$. Thus, if $1 < e < +\infty$, then $\frac{|FV|}{|VE|} > 1$ and, therefore, the conic section is a hyperbola.

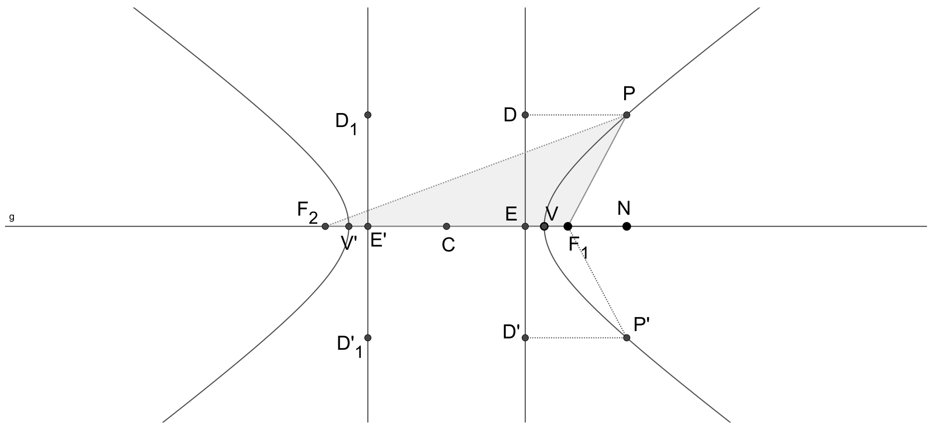


Figure 7: Hyperbola obtained from the construction of a general conic by putting the point V in a position such as $|FV|>|VE|$.

Thus, a hyperbola can be defined as the locus of the points of a plane such that the absolute difference of the distances to two fixed points is constant (Figure 7).

2.2 The Dandelin-Quetélet theorem

In this section we will give an idea of the equivalence of the *focus-directrix* property and the conic section method of the Greeks. We used the references [1] and [2].

According to Eves [2] the relations between the study of the non degenerated conic section and the properties that characterize then as *loci* of points satisfying the focus-directrix properties was established by Quetélet (1796-1874) and Dandelin (1794-1847), in an elegant demonstration of a theorem known as *Dandelin-Quetélet Theorem*, which is stated in a transcript of the original text.

> Si l'on fait movoir dans an cône droit une sphère et que dans une position quelconque de cette dernière, supposée tangente au cône, ou lui mène un plan tangente, l'intersection de ce plan et du cône aura pour foyer le point de contact de la sphère et du plan. [3]

The proof of this theorem is presented in [3].

Germinal Pierre Dandelin was born on April 12, 1794 in France and was a Mathematician, Soldier and Professor of Engineering in Belgium.

Lambert Adolphe Jacques Quetélet was born in Ghent, Belgium on February 22, 1796 and was an astronomer, mathematician, demographer, statistician and sociologist. Quetélet had great interest in mathematics and in 1819 he completed a dissertation on the conic sections entitled *De quibusdam locis geometricis, necnon de curva focal - Of some new properties of the focal distance and some other curves.* In 1820 he became a member of the *Royal Academy.* For more details see [6].

Dandelin and Quetélet met at the Lycée in Ghent and became close friends. Quetélet exerted great influence on Dandelin. Both Dandelin and Quetélet published works with relevant results on the conic sections.

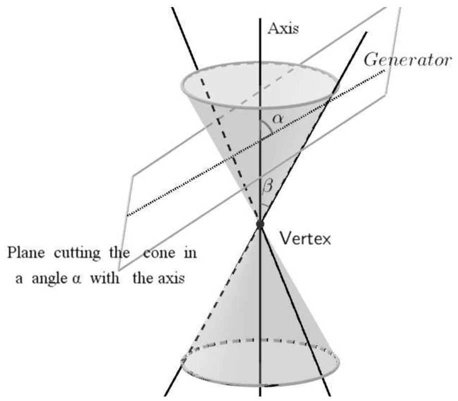
In [3] Dandelin presented an extraordinary characterization of the foci and the directrix of a conic section. He showed that the foci coincide with the intersecting points between two spheres inscribed in the cone and a secant plane that cuts the cone, such as the plane is simultaneously tangent to the both spheres.

However, this result presented by Dandelin was only possible due to a work of Quetélet, in which he observed that the intersection of each sphere with the cone generates a circle, so that the intersections of the planes containing these circles and the plane tangent to the spheres (secant plane) are exactly the directrices of the conic sections. It is worth to observe that if the two spheres are distinct, then the conic section will be an ellipse or a hyperbola, depending on whether both spheres are inscribed in one or two nappes (the two pieces of a cone joined by the vertex) of the cone, respectively. If the two spheres are coincident, the conic section will be a parabola with a single directrix.

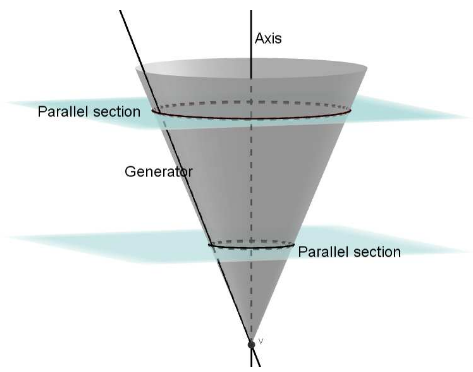
However, Dandelin failed to show the focal property for the parabolas. This property was demonstrated by *Pierce Morton* in 1829, through a construction similar to that of Dandelin. It is important to note that Dandelin and Quetélet used the spheres inscribed in a cone to prove that any conic section is the locus of points for which the ratio between the distance to the focus and the distance to the directrix is constant, however, they did not use the Dandelin spheres to prove the focus-directrix property.

Next, we present the Dandelin-Quetélet theorem in the current language, but for this we will need some definitions and preliminary results.

Let V be a fixed point in a fixed straight line e and h be a straight line passing through the point V. For simplicity we will consider a double right circular cone generated by the rotation of the line h about the line e in an angle β. We will refer to the line h as the *generator*, the line e as the axis and the point V as the vertex of the cone. This cone is represented in Figure 8a.



(a) a double-napped cone generated by the rotation of the line h about the axis e in an angle β. The angle α is the angle between the axis and the secant plane.



(b) Parallel sections of the cone.

Figure 8

Definition 1. *The intersections of a cone with a plane containing its axis, are two straight lines called meridian sections of the cone. These meridian sections are, in fact, two generators of the cone.*

Definition 2. *The intersections of a cone with two parallel planes, orthogonal to its axis, are two circles belonging to the cone called parallel sections of the cone. (See Figure 8b).*

Proposition 1. *Let A be a point not belonging to a plane π, and let B and C be two arbitrary distinct points of π. Let α and β the angles that the line segments AB and AC make with the plane π, respectively. Then $|AB|$ and $|AC|$ are inversely proportional to the sines of the angles that the line segments make with the plane.*

Proof. It is sufficient to observe that $|AB|$ and $|AC|$ are both hypotenuse of right triangles with adjacent catheti in the π plane, as shown in Figure 9. It is easy to see that $|AD| = |AB|\sin\alpha$ and $|AD| = |AC|\sin\beta$, so, $\dfrac{|AB|}{|AC|} = \dfrac{\sin\beta}{\sin\alpha}$. □

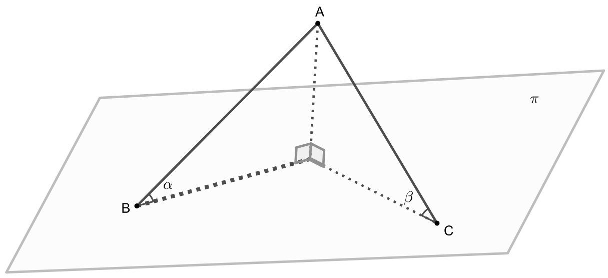


Figure 9: $|AB|$ and $|AC|$ are inversely proportional to the sines of the angles that the line segments make with the plane.

Lemma 1. *Let $S = S(O, r)$ a spherical surface of centre O and radius r. Let PT and PU be tangents to the sphere S. Then the triangles POT and POU are congruents (and therefore $PT = PU$).*

Proof. It is sufficient to observe that the triangles POT and POU are congruent since $|OT| = |OU| = r$, PO is a common side of both two triangles and $\angle OTP = \angle OUP = 90°$ (see Figure 10). □

Corollary 1. *All the line segments determined by the intersections of the meridian sections with two parallel sections of a cone have the same measure.*

Theorem 1. *Let π be a plane that intersects a right circular cone in a conic section, and consider a sphere tangent to the cone and tangent to π at a point F. Let π' be the plane determined by the circle of tangency of the sphere and the cone, and let d be the line of intersection of π and π'. Let P be any point on the conic section, and D be the foot of the line segment from P perpendicular to d. Then the ratio $|\frac{PF}{PD}|$ is a constant.*

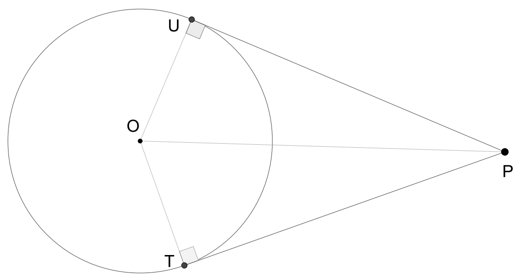


Figure 10: $PT = PU$.

Proof. Let c be the circle of tangency of the sphere and the cone (a parallel section). Let V be the vertex of the cone and P be an arbitrary point on the conic section. Then the line PV is a generator of the cone and therefore intersects the circle c in a point Q. Since the sphere is tangent to the cone and tangent to the plane π at F, then, the lines PF and PQ are both tangents to the sphere, and therefore it follows from Lemma 1 that $|PF| = |PQ|$ (see Figure 11). Let α be the angle between the plane π' and the line PQ and let β denote the angle between π and π'. Then, by the Proposition 1 $|\frac{PF}{PD}| = |\frac{PQ}{PD}| = \frac{\sin\beta}{\sin\alpha}$. So the ratio $|\frac{PF}{PD}|$ is a constant. $\square$

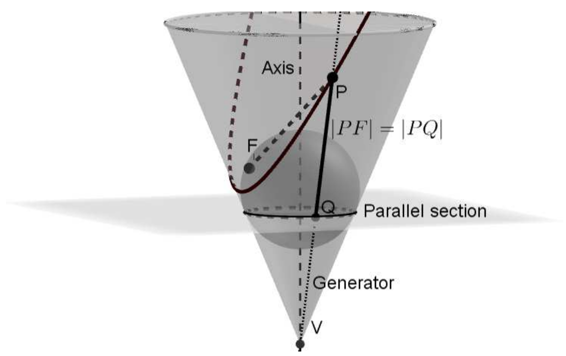


Figure 11: $PT = PU$.

In the proof of the theorem the ratio $|\frac{PF}{PD}|$ is the eccentricity e of the conic section. Thus, if $\sin\beta = \sin\alpha$ we have $e = 1$ and the conic section is a parabola; if $\sin\beta > \sin\alpha$ we have $e < 1$ and the conic section is an ellipse and if if $\sin\beta < \sin\alpha$ we have $e > 1$ and the conic section is a hyperbola.

2.3 Verification of the Dandelin-Quetélet theorem for the ellipse

It follows from Definition 1 that the intersection of a cone and a sphere is a parallel section of the cone.

First note that if the conic section is an ellipse then the secant plane cuts all the generators of the cone, and therefore there exist two distinct spherical surfaces simultaneously inscribed in the cone and tangent to the π plane (one above and the other one below the plane). The intersection of both spheres with the plane determines two circles c_1 and c_2 whose rays are perpendicular to the axis of the cone. Consequently there exist two planes Π and Π' which determine parallel sections c_1 and c_2 of the cone, respectively.

Let F_1 and F_2 be distinct points of tangency between the spheres and the plane π. Observe that $F_1 = F_2$ if and only if the secant plane is orthogonal to the axis, and in this case conic section is a circle.

Take an arbitrary point P at the intersection of the cone with the plane and denote Q the point where the VP generator touches the sphere S_1. Observe that Q is a point of the parallel section c_1. Since VP is tangent to S_1 at the point Q then PQ is tangent to S_1 as well.

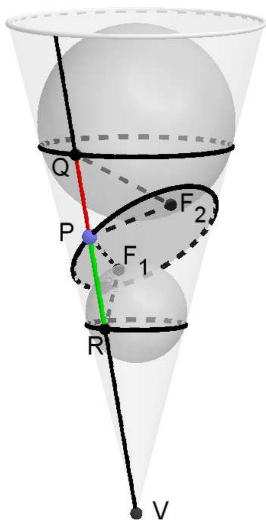


Figure 12: Two Dandelin spheres determine the foci of an ellipse.

On the other hand, F_1 is the contact point of S_1 with the plane π, since P belongs to the intersection of π with the cone, then F_1 and P are both points of π, consequently the line PF_1 is also tangent to S_1 at F_1. So PF_1 and PQ are both tangents to S_1 and, therefore, it follows from Lemma 1 that,

$$|PQ| = |PF_1|. \tag{5}$$

Using this same argument applied to S_2 and F_2 it is shown that

$$|PR| = |PF_2|. \tag{6}$$

Therefore, it follows from (5), (6) and Corollary 1 that $|PF_1|+|PF_2| = |PQ|+|PR| = |QR|$. Since $|QR|$ is a constant, the conic section is an ellipse (Figure 12).

2.4 Verification of the Dandelin-Quetélet theorem for the parabola

The verification of the Dandelin theorem for the parabola is a little more difficult. First we may remember that in the parabola case we have $\alpha = \beta$.

Let π be a plane that intersects a right circular cone of vertex V at an angle α such that $\alpha = \beta$. In this case it is possible to inscribe to the cone a sphere tangent to the plane π at a point F. Let C be the center of this sphere.

Let π_1 be the plane transverse to π determined by the axis of the cone and by the line determined by points F and C. Let c_1 a *parallel section* which determines the contact circumference of the sphere with the cone and let π_2 be the plane containing this parallel section.

Let d be the line given by the intersection of the plane π and the plane π_2, r be the line given by the intersection of the planes π_1 and π_2, and A be the point of intersection between the straight lines r and d.

Let s be the straight line given by the intersection of the planes π and π_1, and let R and Q be the intersection between r and c_1.

Let V' be the intersection of the line s and the generatrix VQ of the cone. Let P be an arbitrary point on the curve of intersection between the plane π and the cone. Let π_3 be a plane containing the point P and parallel to c_1. Observe that π_3 will intercept the line s at a point P', and at the same time, it will intercept the VQ generatrix at a point T .

Also observe that the triangles $V'AQ$ and $V'P'T$ are similar to triangle RVQ (see Figure 13). Since RVQ is isosceles then $V'AQ$ and $V'P'T$ are also isosceles.

Therefore, it follows from the equalities $TQ = TB + V'Q$; $P'A = P'V' + V'A$; $TV' = P'B$ and $V'Q = V'A$, that, $TQ = P'A$ (observe that both P and P' belong to the plane π_3).

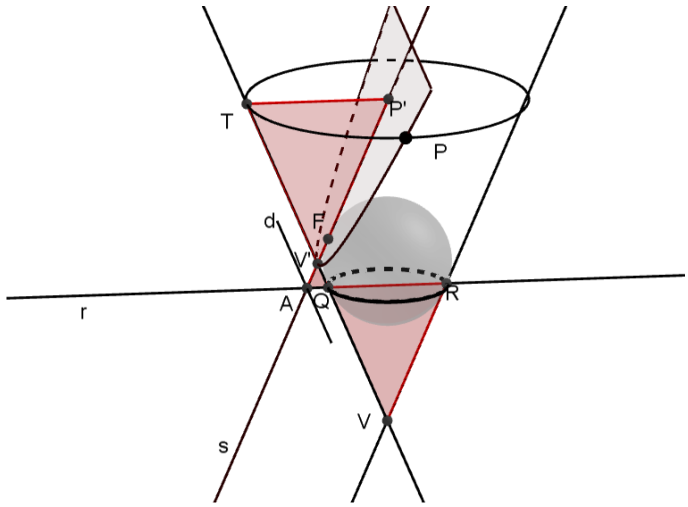


Figure 13: Triangles RVQ, $V'P'T$ and $V'AQ$ are isosceles. Triangles $V'P'T$; $V'AQ$ and $V'P'T$ are similar to triangle RVQ.

On the other hand, it follows from Lemma 1 that

$$PF = PQ = P'A. \tag{7}$$

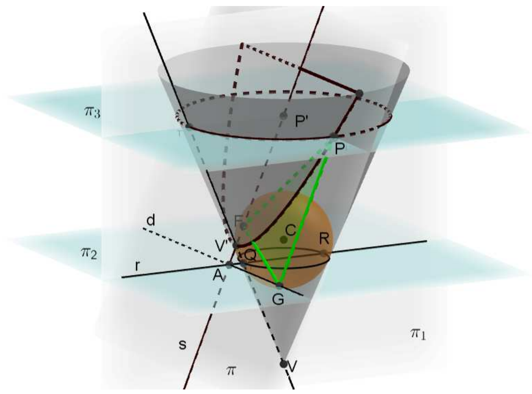
Now consider a point G on the straight line d such that $PG = P'A$ and PG is parallel to $P'A$. It follows from 7 that $PG = P'A = PF$.

Since P is an arbitrary point in the intersection c_2 of the plane π with the cone, then we conclude that, by varying P over c_2, the point G also varies along the straight line d, maintaining the property $d(P, F) = d(P, G)$.

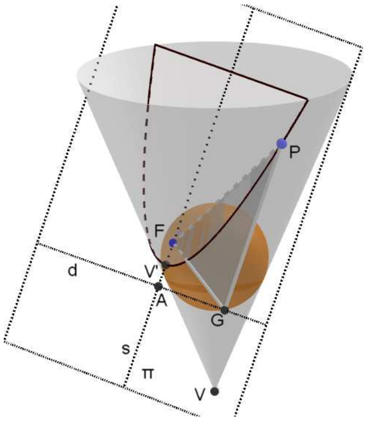
Thus, since $d(P, G)$ is the distance from P to the straight line d, then it can be stated that every point P of the intersection between the cone and the plane π is equidistant of the fixed point F and the fixed line d. The locus of points with this property is a parabola, such as F is the focus, d is the directrix and V' is the vertex, as shown in Figure 14.

The verification of the Dandelin theorem for the hyperbola follows the same idea.

It should be noticed that it is not the goal of this work to bring new results or rigorous demonstrations or even to show connections between the geometric and algebraic definitions. The idea is to establish a relation between the focus-directrix property and the conic section method of the Greeks in a new perspective that enables the reader to perceive the difference between the two methods and to understand both the crucial role of the conic sections in the development of modern knowledge and the extraordinary beauty of the theme.



(a) Figure a



(b) Figure b

Figure 14: Figure a represents all the steps of the construction of a parabola whose focus is tangent to a Dandelin sphere. In Figure b we have the final result of the construction.

References

[1] ALSINA, C., NELSEN, R. B. **A Mathematical Space Odyssey:** Solid Geometry in the 21st Century, Washington, Mathematical Association of America, 2015.

[2] EVES, H. **An Introduction to the History of Mathematics**, 4th ed., New York, Saunders College Publishing 1953; enlarged, 1976.

[3] DANDELIN, G. P. Mémoire sur Quelques Propriétés Remarquables de la Focale Parabolique, **Nouveaux mémoires de l'Académie Royale des Sciences et Belles-Lettres de Bruxelles**, v. 2, 171-202, 1822.

[4] FRIED, M. N. **Edmond Halley's Reconstruction of the Lost Book of Apollonius's Conics.** Translation and Commentary, New York, Springer-Verlag. 2012.

[5] MERZBACH, U. C., Boyer, C. B. **A History of Mathematics**, 3rd ed., New Jersey, John Wiley and Sons, Inc, 2011.

[6] MOSSELMANS, B. **William Stanley Jevons and the Cutting Edge of Economics**, Routledge, London, 2007, Routledge Studies in the History of Economics.

[7] QUETÉLET, L. A. J. Mémoire sur une Nouvelle Theorie des Sections Coniques Considérées Dans le Solide, **Nouveaux mémoires de l'Académie Royale des Sciences et Belles lettres de Bruxelles**, v. 2. p. 123-153, 1822.

[8] TAYLOR, C. **An Introduction to the Ancient and Modern Geometry of Conics**, Deighton, Bell and co., 1881.

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